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Determination of the Number of Common Stochastic Trends under Conditional Heteroskedasticity[1]

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ABSTRACT

Permanent-transitory decompositions and the analysis of the time series properties of economic variables at the business cycle frequencies strongly rely on the correct detection of the number of common stochastic trends (co-integration). Standard techniques for the determination of the number of common trends, such as the well-known sequential procedure proposed in Johansen (1996), are based on the assumption that shocks are homoskedastic. This contrasts with empirical evidence which documents that many of the key macro-economic and financial variables are driven by heteroskedastic shocks. In a recent paper, Cavaliere *et al.*, (2010, *Econometric Theory*) demonstrate that Johansen's (LR) trace statistic for co-integration rank and both its i.i.d. and wild bootstrap analogues are asymptotically valid in non-stationary systems driven by heteroskedastic (martingale difference) innovations, but that the wild bootstrap performs substantially better than the other two tests in finite samples. In this paper we analyse the behaviour of sequential procedures to determine the number of common stochastic trends present based on these tests. Numerical evidence suggests that the procedure based on the wild bootstrap tests performs best in small samples under a variety of heteroskedastic innovation processes.

Keywords: Co-Integration, Maximum Eigenvalue Rank Tests, Conditional Heteroskedasticity, I.I.D. Bootstrap, Wild Bootstrap.

Determinación del número de tendencias estocásticas comunes bajo heteroscedasticidad condicional

RESUMEN

Tanto las descomposiciones en componentes permanentes-transitorias de las series de tiempo como el análisis de las propiedades como tales de las variables económicas en las frecuencias del ciclo económico (business cycle) dependen fuertemente de la detección correcta del número de tendencias estocásticas comunes (cointegración). Las técnicas estándar para la determinación del número de tendencias comunes, como, por ejemplo, el conocido procedimiento secuencial propuesto en Johansen (1996), se basan en la hipótesis de que los shocks son homoscedásticos. Esto contradice la evidencia empírica que demuestra que muchas de las variables financieras y macroeconómicas más importantes se mueven por shocks heteroscedásticos. En un artículo reciente, Cavaliere y otros autores (2010, *Econometric Theory*) demuestran que el estadístico LR de la traza para el rango de la co-integración y sus análogos (tanto los i.i.d. como los "wild" bootstrap) son válidos asintóticamente en sistemas no estacionarios dirigidos por innovaciones heteroscedásticas (diferencia de martingalas) y que, además, "wild bootstrap" funciona sustancialmente mejor que los otros dos contrastes en muestras finitas. En este artículo, basándonos en esta prueba, analizaremos el comportamiento de procedimientos secuenciales para determinar, sobre la base de esos test, el número de tendencias estocásticas comunes presentes. La evidencia numérica sugiere que el procedimiento basado en los test "wild bootstrap" funciona mejor para pequeñas muestras y bajo una variedad de procesos de innovaciones heteroscedásticas.

Palabras clave: Cointegración, test de rango de máximo autovalor, heteroscedasticidad condicional, i.i.d. bootstrap, wild bootstrap.

Clasificación JEL: C30, C32.

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1. INTRODUCTION

Permanent-transitory decompositions and the analysis of the time series properties of economic variables at the business cycle frequencies strongly rely on the correct detection of the number of common stochastic trends (co-integration). Standard techniques for the determination of the number of common trends, such as the well-known sequential procedure proposed in Johansen (1996), are based on the assumption that shocks are homoskedastic. This contrasts with the findings in recent applied work, see e.g. Gonçalves and Kilian (2004), which suggest that the assumption of independent identically distributed (i.i.d.) innovations driving time series models for macro and financial data is untenable in practice. Moreover, as illustrated here in the empirical example reported in Section 6, the presence of conditional heteroscedasticity may also critically affect the decision on the number of common trends when using conventional approaches which do not allow for such behaviour in the data.

In a recent paper Cavaliere *et al.* (2010) [CRT] discuss the behaviour of the trace [LR] test for co-integrating rank of Johansen (1996) under conditional heteroskedasticity. CRT show that the asymptotic LR test, although derived under the assumption of Gaussian i.i.d. innovations, remains asymptotically valid under conditional heteroskedasticity. Moreover, CRT demonstrate the asymptotic validity of both the i.i.d. bootstrap LR test of Swensen (2006) and the corresponding wild bootstrap test which they propose, under conditional heteroskedasticity. Simulation results reported in CRT for a variety of conditionally heteroskedastic innovation models suggest that the wild bootstrap test clearly outperforms both the asymptotic and i.i.d. bootstrap LR tests in finite samples.

In this paper we extend the analysis of CRT to investigate the behaviour of a sequential procedure for determining the number of common stochastic trends of the system based on the wild bootstrap trace statistics. This is compared, for a variety of popular conditionally heteroskedastic models, with the corresponding sequential procedures based on the asymptotic trace statistics and on the i.i.d. bootstrap trace statistics of Swensen (2006). Overall we find that the wild bootstrap-based sequential procedure again delivers significant improvements in finite samples over the performance of both the asymptotic procedure and that based on the i.i.d. bootstrap.

The paper is organized as follows. Section 2 outlines the reference co-integrated VAR model driven by conditionally heteroskedastic (martingale difference) innovations and recaps the trace statistic of Johansen (1996). The analogous wild bootstrap test of CRT is briefly outlined in section 3 along with the corresponding the i.i.d. re-sampling bootstrap rank tests of Swensen (2006). In section 4 we outline the proposed sequential procedure based on the wild bootstrap trace tests. The finite sample properties of this procedure are explored through Monte Carlo methods in section 5 and are compared with those of the corresponding procedures based on the standard asymptotic and i.i.d. bootstrap tests. In section 6 we apply our sequential procedure to US interest rate data, finding evidence consistent with recent multifactor theories of the term structure. Section 7 concludes.

In the following '$\xrightarrow{w}$' denotes weak convergence, '$\xrightarrow{p}$' convergence in probability, and '$\xrightarrow{w}_p$' weak convergence in probability (Giné and Zinn, 1990; Hansen, 1996), in each case as the sample size diverges to positive infinity; $\mathrm{I}(\cdot)$ denotes the indicator function and '$x := y$' ('$x =: y$') indicates that x is defined by y (y is defined by x). The space spanned by the columns of any $m \times n$ matrix A is denoted as $\mathtt{col}(A)$; if A is of full column rank $n < m$, then $A_\perp$ denotes an $m \times (m-n)$ matrix of full column rank satisfying $A_\perp' A = 0$. For any square matrix, A, $|A|$ is used to denote the determinant of A, $\|A\|$ the norm $\|A\|^2 := \mathtt{tr}\{A'A\}$, where $\mathtt{tr}\{A\}$ denotes the trace of A, and $\rho(A)$ its spectral radius (that is, the maximal modulus of the eigenvalues of A). For any vector, x, $\|x\|$ denotes the usual Euclidean norm, $\|x\| := (x'x)^{1/2}$. Finally, $a \,|\, b(\cdot) := a(\cdot) - \int a(s)\, b(s)' ds \left(\int b(s)\, b(s)' ds\right)^{-1} b(\cdot)$ is used to denote the projection residuals of a onto b.

2. THE CONDITIONALLY HETEROSKEDASTIC CO-INTEGRATION MODEL AND ASYMPTOTIC TESTS

Following CRT, we consider the following VAR(k) model in error correction format:

$$\Delta X_t = \Pi X_{t-1} + \Psi U_t + \mu D_t + \varepsilon_t, \quad t = 1,...,T \tag{2.1}$$

where: X_t and ε_t are $p \times 1$, $U_t := (\Delta X_{t-1}', ..., \Delta X_{t-k+1}')'$ is $p(k-1) \times 1$ and $\Psi := (\Gamma_1, ..., \Gamma_{k-1})$, where $\{\Gamma_i\}_{i=1}^{k-1}$ are $p \times p$ lag coefficient matrices and the impact matrix $\Pi := \alpha\beta'$ where α and β are full column $p \times r$ matrices, $r \le p$. The term D_t, which will be discussed in more detail below, collects all deterministic components in the model. The initial values, $X_0 := (X_0', ..., X_{-k+1}')'$, are taken to be fixed.

As in CRT, the process in (1) is taken to satisfy the following two assumptions:

Assumption 1: (a) *all the characteristic roots associated with (2.1); that is of* $A(z) := (1-z)\, I_p - \alpha\beta' z - \Gamma_1 z(1-z) - \cdots - \Gamma_{k-1} z^{k-1}(1-z) = 0$, *are outside the unit circle or equal to 1*; (b) $\det(\alpha_\perp' \Gamma \beta_\perp) \neq 0$, *with* $\Gamma := I_p - \Gamma_1 - \cdots - \Gamma_{k-1}$.

Assumption 2: *The innovations* $\{\varepsilon_t\}$ *form a martingale difference sequence with respect to the filtration* F_t, *where* $F_{t-1} \subseteq F_t$ *for* $t = ..., -1, 0, 1, 2, ...$ *satisfying: (i)*

$$\frac{1}{T}\sum_{t=1}^{T} E(\varepsilon_t \varepsilon_t' \,|\, F_{t-1}) \xrightarrow{p} \Sigma > 0, \tag{2.2}$$

and (ii) $E\|\varepsilon_t\|^4 \le K < \infty$.

Assumption 1 is standard in the co-integration testing literature, while Assumption 2 implies that ε_t is a serially uncorrelated, potentially conditionally heteroskedastic process. The latter therefore contrasts with the assumption that ε_t is i.i.d. as made in Johansen (1996) and Swensen (2006). The rank of Π is the so-called co-integrating rank; under Assumptions 1-2, $p-r$ denotes the number of common stochastic trends.

For unknown parameters α, β, Ψ, μ, and when α and β are $p\times r$ matrices, not necessarily of full rank, (2.1) denotes our conditionally heteroskedastic co-integrated VAR model, which we denote as $H(r)$. The model may then be written in the compact form

$$Z_{0t} = \alpha\beta^{*'} Z_{1t} + \mu_2 Z_{2t} + \varepsilon_t \tag{2.3}$$

with $Z_{0t} := \Delta X_t$, and Z_{1t} and Z_{2t} defined according to the following cases for the deterministic terms, as in Johansen (1996, p. 81):

1. $\mu D_t = 0$ in (2.1), $Z_{1t} := X_{t-1}$ and $Z_{2t} := U_t$ (no deterministic components).
2. $\mu D_t = \mu_1 = \alpha\rho'$ in (2.1), $Z_{1t} := (X'_{t-1}, 1)'$ and $Z_{2t} := U_t$ (restricted constant).
3. $\mu D_t = \mu_1 + \mu_2 t$ with $\mu_2 = \alpha\rho'$ in (2.1), $Z_{1t} := (X'_{t-1}, t)'$ and $Z_{2t} := (U'_t,\ 1)'$ (restricted linear trend).

As is standard, let $M_{ij} := T^{-1}\sum_{t=1}^{T} Z_{it}Z'_{jt}$, $i,j = 0,1,2$, with Z_{it} defined as in (2.3), and let $S_{ij} := M_{ij} - M_{i2}M_{22}^{-1}M_{2j}$, $i,j = 0,1$. Under the assumption of i.i.d. Gaussian disturbances, the pseudo Gaussian likelihood function depends on the vector $\theta^{PML} := (\alpha,\beta,\Psi,\mu,\Sigma)$ (throughout we apply the usual norming or identication as in Johansen, 1996, section 13.2). We denote the corresponding pseudo Maximum Likelihood (PML) estimator as $\hat{\theta}^{PML} := (\hat{\alpha},\hat{\beta},\hat{\Psi},\hat{\mu},\hat{\Sigma})$. Write the maximized (pseudo) log-likelihood under $H(r)$, say $\ell(r)$, as

$$\ell(r) = -\frac{T}{2}\log\left|S_{00}\right| - \frac{T}{2}\sum_{i=1}^{r}\log(1-\hat{\lambda}_i)$$

where $\hat{\lambda}_1 > \ldots > \hat{\lambda}_p$, solve the eigenvalue problem

$$\left|\lambda S_{11} - S_{10}S_{00}^{-1}S_{01}\right| = 0. \tag{2.4}$$

The pseudo LR (PLR) test for $H(r)$ vs $H(p)$ then rejects for large value of the statistic

$$Q_r := -2\left(\ell(r) - \ell(p)\right) = -T\sum_{i=r+1}^{p}\log\left(1-\hat{\lambda}_i\right). \tag{2.5}$$

CRT demonstrate that for data generated according to DGP (2.1) under Assumptions 1 and 2, that under the null hypothesis $H(r)$, the first-order limiting null distribution of Q_r coincides with that given in Johansen (1996) for the homoskedastic case where $\varepsilon_t \sim \text{i.i.d.}(0,\Sigma)$ with finite fourth moments; that is,

$$Q_r \xrightarrow{w} \text{tr}\left(Q_B\right) =: Q_{r,\infty} \tag{2.6}$$

where

$$Q_B := \text{tr}\left(\int (dB(u))\,F(u)'\left(\int F(u)\,F(u)'\right)^{-1} \times \int F(u)\,(dB(u))'\right)$$

where $B(\cdot)$ is a $(p-r)$-variate standard Brownian motion and where F is a function of B whose precise form depends on the deterministic term. More specifically,

1. if $\mu D_t = 0$ in (2.1), then $F := B$;
2. if $\mu D_t = \alpha\rho_1'$ in (2.1), then $F := (B',1)'$;
3. if $\mu D_t = \mu_1 + \alpha\rho_2' t$ in (2.1), then $F := (B', u\,|\,1)'$,

see also Johansen (1996).

3. BOOTSTRAP PLR TESTS

In this section we briefly review the wild bootstrap trace tests of the null hypothesis $H(r)$ against $H(p)$, $r < p$ of CRT and the corresponding i.i.d. bootstrap tests of Swensen (2006). The implementation of both methods requires us only to estimate the VAR (k) model under $H(p)$ (i.e., the unrestricted VAR) and under $H(r)$.

As in section 2, let $\hat{\Psi} := (\hat{\Gamma}_1,...,\hat{\Gamma}_{k-1})$ and, where appropriate, $\hat{\mu}$ denote the PML estimates of Ψ and μ, respectively, from the model under $H(p)$; the corresponding unrestricted residuals are denoted by $\hat{\varepsilon}_t$, $t = 1,...,T$. In addition, let

$\hat{\alpha}^{(r)}, \hat{\beta}^{(r)}$ denote the PML estimates of α, β under the null hypothesis $H(r)$[2]. As discussed in Swensen (2006) and CRT, both the wild and i.i.d. bootstrap algorithms outlined below require that the roots of the equation $\left|\hat{A}^{(r)}(z)\right| = 0$ are either one or outside the unit circle, where

$$\hat{A}^{(r)}(z) := (1-z)\, I_p - \hat{\alpha}^{(r)}\hat{\beta}^{(r)\prime} z - \hat{\Gamma}_1 (1-z)\, z - \cdots - \hat{\Gamma}_{k-1}(1-z)\, z^{k-1}. \qquad (3.1)$$

Moreover, both bootstrap methods require that $\left|\hat{\alpha}_{\perp}^{(r)\prime}\hat{\Gamma}\hat{\beta}_{\perp}^{(r)}\right| \neq 0$ ($\hat{\Gamma} := I_p - \hat{\Gamma}_1 - \cdots - \hat{\Gamma}_{k-1}$). While the latter condition is always satisfied in practice (and, hence, does not need checking), if the former condition is not met, then the bootstrap algorithm cannot be implemented, because the bootstrap samples may become explosive; cf. Swensen (2006, Remark 1).

3.1 CRT's Wild Bootstrap Algorithm

The following steps constitute the wild bootstrap algorithm of CRT:

Algorithm 1 (Wild Bootstrap Co-integration Test)

Step 1: *Generate T bootstrap residuals* ε_t^b, $t = 1,...,T$, *according to the device*

$$\varepsilon_t^b := \hat{\varepsilon}_t w_t$$

where $\{w_t\}_{t=1}^T$ *denotes an independent* $N(0,1)$ *scalar sequence.*

Step 2: *Construct the bootstrap sample recursively from*

$$\Delta X_t^b := \hat{\alpha}\hat{\beta}' X_{t-1}^b + \hat{\Gamma}_1 \Delta X_{t-1}^b + \cdots + \hat{\Gamma}_{k-1}\Delta X_{t-k+1}^b + \varepsilon_t^b, \quad t = 1,...,T, \qquad (3.2)$$

with initial values, $X_{-k+1}^b = \cdots = X_0^b = 0$.

[2] Specifically, and taking the case of no deterministics to illustrate, with the ordered eigenvalues, $\hat{\lambda}_1 > \cdots > \hat{\lambda}_p$, obtained as laid out in section 2, let $\hat{v} := (\hat{v}_1, \hat{v}_2, ..., \hat{v}_p)$ denote the corresponding eigenvectors, which satisfy $\hat{v}' S_{11}\hat{v} = I_p$, and $\hat{v}' S_{10} S_{00}^{-1} S_{01}\hat{v} = \hat{\Lambda}_p := \text{diag}\,(\hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_p)$. For each $r = 0,1,..., p-1$, the (uniquely defined) QMLE, $\hat{\beta}^{(r)}$, is then given by $\hat{\beta}^{(r)} := \hat{v} K_p^{(r)}$, where $K_p^{(r)} := (I_r, 0_{r\times(p-r)})'$, is a selection matrix indexed by r and p. After $\hat{\beta}^{(r)}$ is computed, the estimator $\hat{\alpha}^{(r)}$ is obtained in the usual manner by simple OLS regression (see Johansen, 1996); that is, $\hat{\alpha}^{(r)} = S_{01}\hat{\beta}^{(r)}(\hat{\beta}^{(r)\prime} S_{11}\hat{\beta}^{(r)})^{-1}$.

Step 3: *Using the bootstrap sample,* $\{X_t^b\}$, *obtain the bootstrap test statistic,* $Q_r^b := -2(\ell^b(r) - \ell^b(p))$, *where* $\ell^b(r)$ *and* $\ell^b(p)$ *denote the bootstrap analogues of* $\ell(r)$ *and* $\ell(p)$, *respectively.*

Step 4: *Bootstrap p-values are then computed as,* $p_{r,T}^b := 1 - G_{r,T}^b(Q_r)$, *where* $G_{r,T}^b(\cdot)$ *denotes the conditional (on the original data) cumulative distribution function (cdf) of* Q_r^b.

Step 5: *Approximate the unknown cdf* $G_{r,T}^b(\cdot)$ *required in Step 4 of Algorithm 1 by generating N (conditionally) independent bootstrap statistics,* $Q_{n:r}^b$, $n = 1,...,N$, *computed as above but recursively from*

$$\Delta X_{n:t}^b := \hat{\alpha}\hat{\beta}' X_{n:t-1}^b + \hat{\Gamma}_1 \Delta X_{n:t-1}^b + \cdots + \hat{\Gamma}_{k-1} \Delta X_{n:t-k+1}^b + \varepsilon_{n:t}^b, \quad t = 1,...,T,$$

initialised at $X_{n:-k+1}^b = \cdots = X_{n:0}^b = 0$ *and with* $\{\{w_{n:t}\}_{t=1}^T\}_{n=1}^N$ *a doubly independent* $N(0,1)$ *sequence. The simulated bootstrap p-value is then computed as* $\tilde{p}_{r,T}^b := N^{-1} \sum_{n=1}^N (Q_{n:r}^b > Q_r)$, *and is such that* $\tilde{p}_{r,T}^b \xrightarrow{\text{a.s.}} p_{r,T}^b$ *as* $N \to \infty$. *Note that an asymptotic standard error for* $\tilde{p}_{r,T}^b$ *is given by* $(\tilde{p}_{r,T}^b(1 - \tilde{p}_{r,T}^b)/N)^{1/2}$; *cf. Hansen (1996, p.419).*

CRT establish the asymptotic validity of the wild bootstrap method outlined in Algorithm 1 under conditional heteroskedasticity. In particular they demonstrate the equivalence of the first-order limiting null distributions of the Q_r^b and Q_r statistics, for a given deterministic case. Formally, they show that for data generated according to DGP (2.1) under Assumptions 1 and 2, then under $H(r)$, $Q_r^b \xrightarrow{w}_p Q_{r,\infty}$. A consequence of this is that the bootstrap *p*-values are (asymptotically) uniformly distributed under the null hypothesis, leading to tests with (asymptotically) correct size in the presence of conditional heteroskedasticity of the form given in Assumption 2.

3.2. Swensen's i.i.d. Bootstrap Algorithm

The i.i.d. bootstrap method outlined in Swensen (2006) follows the same steps as the wild bootstrap method outlined above in section Sec 3.1, except that Step 1 of Algorithm 1 is replaced as follows:

Algorithm 2 (i.i.d. Bootstrap Co-integration Test)

Step 1: *Generate T bootstrap residuals* $\varepsilon_t^s,\ t=1,...,T,$ *as independent draws with replacement from the centred residuals* $\{\hat{\varepsilon}_t - T^{-1}\sum_{i=1}^{T}\hat{\varepsilon}_i\}_{t=1}^{T}$.

Steps 2-5: *Follow steps 2-5 of Algorithm 1, replacing* ε_t^b *by* ε_t^s *throughout*[3]. *Denote the resulting i.i.d. bootstrap rank statistic by* Q_r^s *and the associated i.i.d. bootstrap p-value by* $p_{r,T}^s$.

Under the (homoskedastic) assumption that $\varepsilon_t \sim \mathbf{i.i.d.}(\mathbf{0},\boldsymbol{\Sigma})$ with finite fourth moments, Swensen (2006) demonstrates that, under $H(r)$, the i.i.d. bootstrap rank statistic Q_r^s replicates the first-order asymptotic null distribution of the standard trace statistic, Q_r of (2.5). CRT demonstrate that the i.i.d. bootstrap method of Swensen (2006) remains asymptotically valid under the conditionally heteroskedastic conditions of Assumption 2; that is, they show that for data generated according to DGP (2.1) under Assumptions 1 and 2, then under $H(r)$, $Q_r^s \xrightarrow{w}_p Q_{r,\infty}$, and consequently that the bootstrap *p*-values are, as with the corresponding wild bootstrap *p*-values, (asymptotically) uniformly distributed under the null hypothesis.

4. BOOTSTRAP SEQUENTIAL PROCEDURES

The most important use of the likelihood ratio test outlined earlier is as part of a sequential procedure to determine the co-integration rank (i.e., the number of common trends). Johansen (1996) outlines a sequential approach to determining the co-integration rank, based on the sequence of trace statistics, $Q_0, Q_1, ..., Q_{p-1}$. This procedure starts with testing $r = 0$ against $r = p$ using the Q_0 test and sequentially raises r by one until for $r = \hat{r}$ the test statistic $Q_{\hat{r}}$ does not exceed the η level critical value for the test. For the case of i.i.d. errors, Johansen (1996) demonstrates that this procedure is consistent in the sense that the correct co-integrating rank will be selected with probability $(1-\eta)$ in large samples[4]. This result follows because under $H(r)$, the r largest eigenvalues solving (2.4),

[3] In fact, Swensen (2006) suggests including the estimated deterministic component in the right member of (3.2) and initialising at $X_{-k+1}^b = X_{-k+1}, \ldots, X_0^b = X_0$. We do not do this because, as shown in Caveliere, Taylor and Trenkler (2010), this yields tests which are not invariant to the level of the deterministic component, such that their finite sample size and power will depend on the latter.

[4] If the true rank is p then this will be selected with probability 1 in large samples.

$\hat{\lambda}_1,\ldots,\hat{\lambda}_r$, converge in probability to positive numbers, while $T\hat{\lambda}_{r+1},\ldots,T\hat{\lambda}_p$ are of $O_p(1)$. Consequently, the PLR test based on Q_r will be consistent at rate $O_p(T)$ if the true co-integration rank is, say, $r_0 > r$. CRT argue that the same result holds true under conditional heteroskedasticity of the form given in Assumption 2.

The issue of how to implement a bootstrap version of Johansen's sequential approach is not trivial. For instance, Swensen (2006, p.1700) observes that

> "*This problem is more intriguing and presents some new aspects that are nonstandard in a bootstrap context, because we have to do the resampling for different values of the rank of the estimated reduced rank matrix. The dimension of the cointegration space in the generated observations will therefore not correspond to the true cointegration rank, but to the imposed rank* (…)".

As in Algorithm 2 of Swensen (2006) and in Algorithm 1 of CRT, the sequential bootstrap procedure is implemented by estimating α, β (together with ρ in the case of deterministic terms) under $H(r)$, and then using these estimated values in the re-sampling recursion equation. As discussed in Swensen (2009), this throws up a potential complication in those steps of the procedure where the null rank being tested is smaller than the true co-integrating rank. To be valid here, the additional conditions outlined in Assumption 2 of Swensen (2009) must be assumed to hold.

In order to describe the bootstrap sequential procedure, as before we denote by $\hat{\Psi} := (\hat{\Gamma}_1,....,\hat{\Gamma}_{k-1})$ the estimates of $\Psi := (\Gamma_1,....,\Gamma_{k-1})$ obtained from the unrestricted model; i.e. those obtained setting $r = p$. At each stage of our proposed sequential algorithm, the bootstrap trace statistic is calculated from re-sampled data which are constructed using $\hat{\Psi}$ and the (Gaussian) (quasi-) likelihood-based estimates [QMLE] of α and β under $H(r)$, the latter two denoted by $\hat{\alpha}^{(r)}$ and $\hat{\beta}^{(r)}$. Notice, therefore, that this model will be misspecified unless *r* is the true rank.

Using these restricted estimators and defining $\hat{A}^{(r)}(z)$ as in (3.1), the bootstrap sequential procedure outlined in Swensen (2006) and CRT is then as follows.

Algorithm 3 (Sequential Bootstrap Algorithm)

Starting from $r = 0$, *perform the following steps:*

Step i: *Check whether the roots of the equation* $\left|\hat{A}^{(r)}(z)\right| = 0$ *are either one or outside the unit circle. If this is not the case, the algorithm stops.*

Step ii: *Generate T bootstrap errors* ε_t^* *using the residuals from the unrestricted model (i.e., estimated under rank p) using either: (a) the wild bootstrap, so that*

$\varepsilon_t^* := \varepsilon_t^b$ *(see Step 1 of Algorithm 1 above), or (b) the i.i.d. bootstrap, so that* $\varepsilon_t^* := \varepsilon_t^s$ *(see Step 1 of Algorithm 2 above).*

Step iii: *Construct the bootstrap sample recursively from*

$$\Delta X_{r,t}^* := \hat{\alpha}^{(r)}\hat{\beta}^{(r)'}X_{r,t-1}^* + \hat{\Gamma}_1\Delta X_{r,t-1}^* + \cdots + \hat{\Gamma}_{k-1}\Delta X_{r,t-k+1}^* + \varepsilon_t^*, \quad t = 1,...,T,$$

initialised at $X_{-k+1}^* = \cdots = X_0^* = 0.$

Step iv: *Using the bootstrap sample,* $\{X_{r,t}^*\}$, *obtain the bootstrap test statistic,* $Q_r^* := -2(\ell^*(r) - \ell^*(p))$, *where* $\ell^*(r)$ *and* $\ell^*(p)$ *denote the bootstrap analogues of* $\ell(r)$ *and* $\ell(p)$, *respectively, along with the corresponding p-value* $p_{r,T}^* := 1 - G_{r,T}^*(Q_r)$ *where* $G_{r,T}^*(\cdot)$ *denotes the conditional (on the original data) cdf of* Q_r^*. *In practice, as detailed in Step 5 of Algorithm 1, the unknown cdf,* $G_{r,T}^*(\cdot)$, *may be numerically approximated using N bootstrap replications.*

Step v: *If* $p_{r,T}^*$ *exceeds the significance level,* η, *set* $\hat{r} = r$, *otherwise repeat steps (i)-(iv) testing the null of rank* $(r+1)$ *against rank p if* $r+1 < p$, *or set* $\hat{r} = p$ *if* $r+1 = p$.

Using the results in CRT, it is straightforward to shown that under the conditions stated in Swensen (2006, 2009) and CRT this sequential procedure is consistent in the sense that correctly selects the true co-integrating rank with probability $(1-\eta)$ in large samples (η denoting the nominal significance level used in each test in the procedure) in the presence of conditional heteroskedasticity satisfying Assumption 2. This property is formalized in the following theorem. The proof is entirely straightforward given the results in CRT and is omitted in the interests of brevity.

Theorem 1. *Let* X_t *be generated according to DGP (2.1) under Assumptions 1 and 2. Further let Assumption 2 of Swensen (2009) hold. Denoting the estimator of the co-integration rank obtained from Algorithm 3 by* $\hat{r}$, *then*

$$\lim_{T\to\infty} P(\hat{r} = r) = 0 \quad \text{for all } r = 0,1,...,r_0 - 1$$
$$\lim_{T\to\infty} P(\hat{r} = r_0) = 1 - \eta \cdot \mathrm{I}(r_0 < p)$$
$$\lim_{T\to\infty} \sup_{r\in\{r_0+1,...,p\}} P(\hat{r} = r) \le \eta.$$

The consistency result in Theorem 1 holds for both the wild bootstrap and i.i.d. based bootstrap sequential procedures of CRT and Swensen (2006, 2009), respectively. A comparison of the finite sample behaviour (together with that of the asymptotic sequential procedure of Johansen, 1996) in the presence of a variety of conditionally heteroskedastic innovation processes is explored numerically in the next section.

5. FINITE SAMPLE SIMULATIONS

In this section we use Monte Carlo simulation methods to compare the finite sample properties of the sequential approach of Johansen (1996) when applied using the asymptotic PLR tests of Johansen (1996) and the wild and i.i.d. bootstrap analogues of the PLR tests of CRT and Swensen (2006), respectively. The simulation model we consider allows for conditional heteroskedasticity in the innovation process driving the VAR model.

In sections 5.1, and 5.2 we follow Johansen (2002) and Swensen (2006), and consider as our simulation DGP an $I(1)$, possibly co-integrated, VAR(1) process of dimension p. We allow the dimension of the VAR process to vary over $p = 2,...,5,$ and consider both the case of no co-integration ($r = 0,$ and hence p common trends) [section 5.1], and of a single co-integrating vector ($r = 1,$ and hence $p-1$ common trends) [section 5.2]. In section 5.3 we will subsequently report results for $r = 0$ in a VAR(2) model, thereby also investigating the finite sample impact of higher-order serial correlation.

The DGP considered in section 5.1 is the multivariate martingale process,

$$\Delta X_t = \varepsilon_t,$$

while a generalisation of this DGP to the non-co-integrated VAR(2) case is detailed in section 5.3. In section 5.2, the DGP is the co-integrated VAR(1) model

$$\Delta X_t = \alpha\beta' X_{t-1} + \varepsilon_t$$

where α and β are $p \times 1$ vectors. In each case $\varepsilon_t := (\varepsilon_{1,t},...,\varepsilon_{p,t})'$ is a p-dimensional martingale difference sequence with respect to the filtration $F_t := \sigma(\varepsilon_t, \varepsilon_{t-1},...)$. Following van der Weide (2002), we assume that ε_t may be written as the linear map

$$\varepsilon_t = \Lambda e_t \tag{5.1}$$

where Λ is an invertible $p \times p$ matrix which is constant over time, while the p components of $e_t := (e_{1,t},...,e_{p,t})'$ are independent across $i = 1,...,p.$ In the case

where the individual components follow a standard GARCH(1,1) process (as is the case with Models A and B below), van der Weide (2002) refers to ε_t as a GO−GARCH(1,1) process. We set $\Lambda = I_p$ in the simulations, without loss of generality[5]. Moreover, in the $r = 1$ case considered in section 5.2, we follow Johansen (2002) and Swensen (2006) by considering DGPs with $\beta := (1,0,...,0)'$ and $\alpha := (a_1, a_2, 0,...,0)'$. This leads to the model

$$\Delta X_{1,t} = a_1 X_{1,t-1} + \varepsilon_{1,t}$$
$$\Delta X_{2,t} = a_2 X_{1,t-1} + \varepsilon_{2,t}$$
$$\Delta X_{i,t} = \varepsilon_{i,t}, \quad i = 3,..., p.$$

In our reported simulations we set $a_1 = a_2 = -0.4$, as in Swensen (2006, Table 2).

Within the context of (5.1) we consider for the individual components of e_t the univariate innovation processes and parameter configurations used in Section 4 of Gonçalves and Kilian (2004), to which the reader is referred for further discussion. These are as follows:

- **Model A** is a standard GARCH(1,1) process driven by standard normal innovations of the form $e_{it} = h_{it}^{1/2} v_{it}$, $i = 1,..., p$, where v_{it} is i.i.d. $N(0,1)$, independent across i, and $h_{it} = \omega + d_0 e_{it-1}^2 + d_1 h_{it-1}$, $t = 0,...,T$. Results are reported for $(d_0, d_1) \in \{(0.5,0.0),(0.3,0.65),(0.2,0.79),(0.05,0.94)\}$.
- **Model B** is the same as Model A except that the v_{it}, $i = 1,..., p$, are independent i.i.d. t_5 (normalised to unit variance) variates.
- **Model C** is the exponential GARCH(1,1) (EGARCH(1,1)) model of Nelson (1991) with $e_{it} = h_{it}^{1/2} v_{it}$, $\ln(h_{it}) = -0.23 + 0.9 \ln(h_{it-1}) + 0.25[|v_{it-1}^2| - 0.3 v_{it-1}]$, with $v_{it} \sim$ i.i.d. $N(0,1)$, independent across $i = 1,..., p$.
- **Model D** is the asymmetric GARCH(1,1) (AGARCH(1,1)) model of Engle (1990) with $e_{it} = h_{it}^{1/2} v_{it}$, $h_{it} = 0.0216 + 0.6896 h_{it-1} + 0.3174[e_{it-1} - 0.1108]^2$, with $v_{it} \sim$ i.i.d. $N(0,1)$, independent across $i = 1,..., p$.
- **Model E** is the GJR−GARCH(1,1) model of Glosten *et al.* (1993) with $e_{it} = h_{it}^{1/2} v_{it}$, $h_{it} = 0.005 + 0.7 h_{it-1} + 0.28[|e_{it-1}| - 0.23 e_{it-1}]^2$, with $v_{it} \sim$ i.i.d. $N(0,1)$, independent across $i = 1,..., p$.

[5] By definition, the PLR statistic does not depend on the matrix Λ, as the eigenvalue problem in (2.4) has the same eigenvalues upon re-scaling (as can be seen by simply pre- and post-multiplying by Λ^{-1} in (2.4)).

- **Model F** is the first-order AR stochastic volatility model: $e_{it} = v_{it} \exp(h_{it})$, $h_{it} = \lambda h_{it-1} + 0.5\xi_{it}$, with $(\xi_{it}, v_{it}) \sim$ i.i.d. $N(0, \text{diag}(\sigma_\xi^2, 1))$, independent across $i = 1,..., p.$. Results are reported for $(\lambda, \sigma_\xi) = \{(0.936, 0.424), (0.951, 0.314)\}$.

The reported simulations were programmed using the **rndKMn** function of Gauss 7.0. All experiments were conducted using 10,000 replications. The sample sizes were chosen within the set $\{50, 100, 200\}$ and the number of replications used in the wild bootstrap algorithm was set to 399. All tests were conducted at the nominal 0.05 significance level. For the reasons outlined in Nielsen and Rahbek (2000) relating to similarity with respect to initial values, the VAR model was fitted with a restricted constant (i.e. deterministic case (ii) outlined under (2.3)), when calculating all of the tests. For the standard PLR tests we employed asymptotic critical values as reported in Table 15.2 of Johansen (1996).

5.1. The Non-Co-Integrated Model ($r = 0$)

Tables 1, 2, 3 and 4, at the end of the paper, report for p = 2, 3, 4 and 5, respectively, the properties of the sequential procedure of Johansen (1996) applied using the Q_r, Q_r^b and Q_r^s $(r = 0,..., p-1)$ tests (as described at the start of section 4 and in Algorithm 3, with significance level η = 0.05) in the column blocks headed Q-based, Q^b-based and Q^s-based, respectively, in the case where the true co-integrating rank is zero.

Since all of the tests were run at the 5% significance level, each of the standard and bootstrap sequential procedures should, in the limit, select $r = 0$ with probability 95% and $r > 0$ with (combined) probability 5%.

Under constant conditional variances (the cases where $d_0 = d_1 = 0$ in Models A and B) it can be seen from the first two panels of Tables 1-4 that both the procedures based on Q^b and Q^s select the correct rank more often than the standard sequential asymptotic procedure (Q_0); for example, in the case of Model A for $p = 5$, while the standard sequential procedure selects the true rank 91.1% of the time for $T = 100$, the corresponding wild and i.i.d. bootstrap procedures selecting the true rank 95.6% and 95.1% of the time, respectively.

When the innovation process displays conditional heteroskedasticity, the benefits of the wild bootstrap sequential procedure over the other procedures become clear. The results in Tables 1-4 show that both the Q and Q^s procedures tend to overestimate the true rank, even for samples as large as $T = 200$, in the presence of conditional heteroskedasticity. In contrast, the performance of the wild bootstrap sequential procedure, Q^b, seems largely satisfactory throughout.

The performances of the Q and Q^s procedures are generally worse, other things being equal, the higher is the VAR dimension, p. For example, in the case of Model A with $d_0 = 0.3$, $d_1 = 0.65$ and $T = 200$, the Q and Q^s procedures select the true rank 90% and 90.7% of the time, respectively, for $p = 2$ (see Table 1), decreasing to 86.1% and 89.1%, respectively, for $p = 5$ (see Table 4). In contrast, here the Q^b procedure select the true rank 94.4% and 94.3% of the time for $p = 2$ and $p = 5$, respectively. The precise model of conditional heteroskedasticity can also make quite a substantial difference to the properties of the sequential procedures. For example, comparing the results for Models A and B, we see that t_5 innovations tend to cause rather less overestimation of the true rank than is seen for standard normal innovations. Of all the models considered, it is the autoregressive stochastic volatility case, Model F, which appears to have the strongest impact on the performance of the sequential procedures. The two parameter configurations both imply relatively strong serial dependence in the conditional variance of the innovation process. Here the percentage of time the standard PLR sequential procedure, Q_0, selects the true rank is between around 60% to 80% depending on p and the parameter configuration, while the i.i.d. bootstrap procedure, Q_0^s, performs only slightly better. For example, under Model F for the first parameter configuration and $p = 5$ the procedures based on the standard and i.i.d. bootstrap PLR tests select the correct co-integrating rank only 62.9% and 69.2% of the time, respectively, even for $T = 200$; indeed, each will wrongly indicate that the true co-integrating rank is two about 5% of the time. In contrast, the procedure based on the wild bootstrap PLR tests appears to perform very well in practice, with its empirical probability of selecting the true co-integrating rank of zero converging rapidly towards 95% throughout. In the same example as above, the wild bootstrap-based procedure selects the true co-integrating rank 92.1% of the time, and a rank of two only around 1% of the time.

Although the wild bootstrap procedure, Q^b, tends to overestimate the true rank under Model F, it still represents an enormous improvement on the performance of the other procedures. Moreover, the performance of the wild bootstrap procedure improves, other things equal, as the sample size is increased. Notice that this last observation is not the case for the Q and Q^s procedures where the degree of overestimation *increases* as the sample size increases. Very significant over estimation, although not as bad as for Model F, is also seen for the Q and Q^s procedures in each of Models C, D and E. Again here the wild bootstrap procedure is much better behaved throughout.

In summary, we see that the procedure based on the wild bootstrap gets considerably closer to the nominal level 95% in small samples than do the procedures based on the standard and i.i.d. bootstrap tests, the latter two tending to perform worse the higher is p. Indeed these latter two procedures can perform very poorly indeed under conditional heteroskedasticity.

5.2. The Co-Integrated Model ($r = 1$)

Tables 5,6,7 and 8, at the end of the paper, report results for the sequential procedure of Johansen (1996) for each of the three tests for $p = 2, 3, 4$ and 5, respectively, when the true co-integrating rank is one. Again the procedures based on the Q_r, Q_r^b and Q_r^s $(r = 0,...,p-1)$ tests are reported in the column blocks headed Q-based, Q^b-based and Q^s-based, respectively.

Since now the co-integrating rank is one, these procedures should, in the limit, select $r = 0$ with probability 0%, $r = 1$ with probability 95% and $r > 1$ with (combined) probability 5%. While these proportions are largely maintained, at least for $T = 200$, by the wild bootstrap-based procedure, the same cannot be said for the procedures based on the standard PLR and i.i.d. bootstrap PLR tests, which as with the corresponding results in Tables 1-4 can display a strong tendency to over-estimate the co-integrating rank under conditional heteroskedasticity, even for quite large sample sizes. It is also interesting to also note that in the smaller sample sizes considered the standard and i.i.d. bootstrap-based procedures display a lesser tendency to under-estimate the true co-integration rank than the wild bootstrap-based procedure: for example, when $p = 3$ and $T = 50$, under Model D (see Table 6) the procedure based on the standard PLR tests selects a co-integrating rank of zero 26.6% of the time, while the wild bootstrap procedure does so 40.1% of the time. This result is, of course, an artefact of the uncontrolled size of the standard Q_0 test, this test in fact having size of 14.4% in this case; cf. Table 2.

5.3. The Non-Co-Integrated VAR(2) Model

To conclude this section, and following Johansen (2002, p.1940), we report some additional results investigating the finite sample behaviour under the null hypothesis of tests for $\Pi = 0$ (so that the true co-integrating rank is zero) in the VAR(2) model

$$\Delta X_t = \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \varepsilon_t$$

with $\Gamma_1 = \xi I_p$, $-1 < \xi < 1$. This model is an interesting extension of the conditionally heteroskedastic VAR(1) model considered in sections 5.1 and 5.2 because it allows for higher-order stationary serial correlation. To that end we set $\xi = 0.5$, which allows for a moderate degree of stationary serial correlation in the process. As regards the innovation term, ε_t, we again considered each of Models A-F, reporting results for a subset of the parameter configurations reported for Models A, B and F in sections 5.1 and 5.2[6]. A restricted constant was again

[6] This was done in the interests of space, the additional results qualitatively adding very little to what is reported.

included in the estimated model. Tables 9, 10, 11, 12, at the end of the paper, report the results for $p = 2, 3, 4, 5,$ respectively.

As before, the procedures based on the Q_r, Q_r^b and Q_r^s $(r = 0,..., p-1)$ tests are reported in the column blocks headed Q-based, Q^b-based and Q^s-based, respectively.

In general, it can be seen that higher-order stationary serial correlation tends to inflate the number of times the standard PLR sequential procedure, Q, select the true rank 0, relative to the corresponding results for the VAR(1) case in Table 1. This is true in both the conditionally homoskedastic and conditionally heteroskedastic cases. To illustrate, for $p = 4$ (see Table 11) in the i.i.d. innovations case (Model A with $d_0 = d_1 = 0$) the Q procedure selects the true rank 58.5% of the time for $T = 50,$ increasing to 89.1 % for $T = 200,$ as compared to 91.3% and 93.5% respectively for the VAR(1) case in Table 3. Both bootstrap procedures also display a tendency to overestimate the true rank for $T = 50$ in this case, but the degree of over-estimation is much smaller than for the asymptotic procedure (the true rank is selected 91.5% of the time for Q^b and 91.1% of the time for Q^s) and are all but eliminated by $T = 200.$ As a second example, under Model C for $p = 5$ (see Table 12) the asymptotic procedure selects the true rank 27.0% for $T = 50$ (77.5% for $T = 200$) compared with 80.0% (85.3%) in the corresponding VAR(1) model (Table 4). Again the higher-order serial correlation does affect the performance of both bootstrap procedures, but again this is to a much lesser extent than for the Q_0 test: in the last example, the true rank is selected by the Q^b and Q^s bootstrap procedures 87.5% and 84.4% (93.5% and 89.8% for $T = 200),$ respectively, compared to 92.9% and 89.9% (94.3% and 88.8% for $T = 200),$ respectively, in Table 4. Overall, both bootstrap procedures cop much better with higher-order serial correlation than does the asymptotic procedure.

As with the results in Tables 1-4 for the VAR(1) case, in the VAR(2) case the results in Tables 9-12 show that the wild bootstrap procedure Q^b is again much better behaved than either the Q or the Q^s procedures in the presence of conditional heteroskedasticity.

6. EMPIRICAL ILLUSTRATION

In this section we illustrate the methods discussed in this paper with a short application to the term structure of interest rates; see Campbell and Shiller (1987) for an early reference. According to classical theory, aside from a constant or stationary risk premium, long-term interest rates are an average of current and expected future short term rates over the life of the investment. Hence, provided

interest rates are well described as $I(1)$ variables, bond rates at different maturities should be driven by a single common stochastic trend, with the spreads between rates at different maturities being stationary. Although early studies tend to corroborate this view, see, for example, Hall *et al.* (1992), more recent research, based on broader sets of maturities, suggests that yields are better characterised by *more* than a single common trend, reflecting possible non-stationarities in the risk premia and additional risk factors, such as the slope and curvature of the yield curve; see, for example, the discussion in Giese (2006).

We consider monthly data of U.S. treasury zero-coupon yields, say $R_t^{(n)}$ where *n* denotes the maturity, with *n* = 1 (one-month), 3 (three-months), 12 (one year), 24 (two years) and 60 (5 years). The sample data cover the period 1970:1-2000:12, thereby considerably extending the 1970:1-1988:12 sample used by Hall *et al.* (1992); see Giese (2006) for further details on the data. Yield levels are displayed in the upper panel of Figure 1 (see the last page of the paper), with the corresponding first differences displayed in the second panel of the figure.

Let $X_t := (R_t^{(1)}, R_t^{(3)}, R_t^{(12)}, R_t^{(24)}, R_t^{(60)})'$. As is standard, we fit a VAR model for X_t with restricted intercept; that is, $D_{2t} = 0$ and $D_{1t} = 1$ in (2.3). The VAR is estimated using Gaussian (quasi-)maximum likelihood under the assumption of constant conditional volatility; cf. Section 3. The number of lags is set to $k = 4$.

All five estimated equations display strong residual heteroskedasticity. Specifically, White's test failed at the 1% level and single equation ARCH LM tests all failed at the 1% level[7]. In addition, the final panel of Figure 1 reports the estimated *variance profiles* of the five unrestricted residual series from the estimated VAR (4) model. That is,

$$\hat{\eta}_i(u) := \frac{\sum_{t=1}^{|Tu|} \hat{\varepsilon}_{it}^2}{\sum_{t=1}^{T} \hat{\varepsilon}_{it}^2}, \; u \in [0,1]$$

for each $i = 1,...,5$; see Cavaliere and Taylor (2007, 2008a,b) for further details. Should ε_{it} have constant variance, the corresponding estimate of the variance profile should approximately follow the 45° line. A quick inspection of Figure 1, however, suggests that this does not seem to be the case here, with the estimated variance profiles reflecting the increased variability in month-to-month yield changes observed in the late 1970s and early 1980s. Furthermore, according to the tests proposed in Cavaliere and Taylor (2008b), the deviations of the estimated variance profiles from the 45° line are all statistically significant indicating the presence of changing variances in the data. Notably, the shape of the estimated variance profiles are also consistent with the findings of Hansen (2003) who argues for the presence of two shifts (the first in September 1979 and the second in

[7] Results are available upon request.

October 1982) in the covariance matrix of a system containing monthly U.S. treasury zero-coupon yields with maturities of 1, 3, 6, 9, 12, 60 and 84 months measured over the period 1970: 1-1995: 12.

In summary, although the computation of the trace statistics is based on constant conditional volatility, the sequential testing procedure used to determination the co-integration rank should take this heteroskedesastic behaviour into account. Consequently, the inferences drawn from the sequential procedure based on the *wild* bootstrap would be expected to be the most reliable of the three procedures in finite samples.

Table 13, at the end of the paper, reports the results of the standard asymptotic sequential procedure of Johansen (1996). Along with this procedure, we report results obtained using Swensen's iid bootstrap sequential procedure as well as the wild bootstrap sequential procedure of CRT. For the standard tests *p*-values were computed as suggested in MacKinnon, Haug and Michelis (1999), while the bootstrap *p*-values were calculated as detailed in Algorithm 3 using 999 bootstrap replications in each case.

The standard sequential procedure detects four co-integrating relations. The results obtained by using the i.i.d. sequential procedure of Swensen (2006) are in accord with this result: while $r = 3$ (two common stochastic trends) is rejected at the 5% significance level, the hypothesis $r = 4$ (one single common stochastic trend) is not rejected at any conventional significance level. While these results therefore appear to corroborate the traditional view of the expectation hypothesis of the term structure, the wild bootstrap *p*-values reported in Table 13, which remain valid in the presence of the apparent heteroskedasticity in the data discussed above, suggest that the evidence of a single common trend underlying the term structure is much weaker. Using the wild bootstrap, the presence of two common stochastic trends ($r = 3$) against the alternative of a single common stochastic trend ($r = 4$) is not rejected, even at the 10% significance level. Hence, there is now a clear indication of two rather than one common trends underlying the dynamics of the five yields considered. This result, which again is in line with the findings of Giese (2006), consequently provides further support in favour of recent multifactor theories of the term structure; see, for example, Diebold, Ji and Li (2007).

7. CONCLUSIONS

In this paper we have discussed methods, based on the well-known sequential procedure proposed in Johansen (1996), for determining the number of common trends in a vector non-stationary time series driven by conditionally heteroskedastic innovations. Existing approaches make use of the asymptotic PLR tests of Johansen (1996) or the i.i.d. bootstrap PLR tests of Swensen (2006). We have discussed a corresponding procedure based on the wild bootstrap PLR tests of Cavaliere *et al.* (2010). Numerical evidence presented suggests that the wild

bootstrap based sequential procedure displays superior finite sample behaviour to the extant procedures under a variety of conditionally heteroskedastic innovation processes. An empirical application to US interest rate data was also reported which, when allowing for heteroskedastic behaviour in the data, suggested the presence of more than one common trend in bond yields over different maturities, consistent with recent multi-factor theories of the term structure.

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ANEXOS

TABLE 1

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =2, True Rank is 0.

			Q-based			Q^b-based			Q^s-based		
		$r =$	0	1	2	0	1	2	0	1	2
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$											
d_0	d_1	T									
0.0	0.0	50	93.7	5.5	0.8	94.3	4.6	1.1	95.1	3.9	1.0
		100	94.7	5.0	0.4	95.1	4.2	0.6	95.4	4.1	0.5
		200	94.7	5.0	0.3	95.4	3.9	0.7	95.2	4.3	0.5
0.5	0.0	50	90.1	9.2	0.7	92.8	6.1	1.1	92.1	6.8	1.1
		100	92.7	6.7	0.6	94.7	4.6	0.7	93.5	5.7	0.8
		200	93.4	6.0	0.6	95.2	4.1	0.6	94.1	5.2	0.7
0.3	0.65	50	89.8	9.1	1.0	93.2	5.5	1.2	91.7	7.0	1.3
		100	90.1	9.1	0.7	94.4	5.0	0.6	91.5	7.4	1.1
		200	90.0	9.1	0.9	94.4	4.9	0.7	90.7	7.9	1.4
0.2	0.79	50	90.7	8.3	1.0	93.4	5.4	1.2	92.4	6.4	1.2
		100	90.1	9.1	0.9	94.4	4.8	0.8	91.3	7.5	1.3
		200	89.2	9.7	1.0	94.5	4.9	0.6	89.9	8.6	1.5
0.05	0.94	50	93.5	5.8	0.7	94.1	4.7	1.3	94.8	4.1	1.2
		100	94.2	5.5	0.3	95.1	4.1	0.8	94.8	4.7	0.6
		200	93.5	6.1	0.4	94.9	4.5	0.7	94.1	5.1	0.8
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}$, $v_{i,t} \sim$ i.i.d. t_5, $i = 1, ..., p$											
d_0	d_1	T									
0.0	0.0	50	93.4	6.0	0.6	94.8	4.2	1.1	94.9	4.2	1.0
		100	94.3	5.4	0.3	95.1	4.3	0.7	95.0	4.4	0.6
		200	94.5	5.0	0.5	95.3	4.1	0.6	95.0	4.2	0.8
0.5	0.0	50	91.5	7.7	0.8	94.0	4.8	1.2	93.2	5.8	1.0
		100	92.7	6.9	0.4	94.7	4.6	0.7	93.6	5.6	0.8
		200	93.5	6.1	0.5	95.0	4.4	0.6	94.4	4.8	0.8
0.3	0.65	50	91.3	8.0	0.7	94.2	4.7	1.1	92.9	6.1	1.0
		100	92.5	7.0	0.5	94.9	4.5	0.6	93.5	5.6	0.9
		200	92.8	6.5	0.6	94.8	4.6	0.7	93.4	5.7	0.9
0.2	0.79	50	92.0	7.3	0.8	94.4	4.5	1.1	93.6	5.3	1.1
		100	92.8	6.7	0.5	94.6	4.7	0.7	93.8	5.5	0.7
		200	92.9	6.5	0.6	95.0	4.4	0.6	93.8	5.5	0.8
0.05	0.94	50	93.1	6.2	0.7	94.8	4.2	1.0	94.7	4.4	0.9
		100	94.1	5.5	0.4	95.0	4.4	0.7	94.7	4.5	0.7
		200	94.2	5.3	0.5	95.3	4.0	0.7	94.6	4.5	0.8
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $\ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}]$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$											
		50	89.0	9.8	1.1	92.8	6.1	1.1	90.8	7.7	1.5
		100	89.7	9.4	0.9	94.4	4.9	0.7	90.8	8.0	1.2
		200	90.3	9.0	0.8	94.7	4.5	0.7	90.9	7.9	1.2
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$											
		50	88.2	10.6	1.2	93.2	5.5	1.3	90.1	8.2	1.7
		100	87.3	11.7	1.1	93.8	5.4	0.8	88.5	9.8	1.6
		200	86.1	12.6	1.3	94.4	4.9	0.7	87.0	11.1	2.0
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$											
		50	89.3	9.6	1.2	93.3	5.6	1.2	91.0	7.6	1.5
		100	88.9	10.1	1.0	94.3	5.1	0.7	90.0	8.8	1.2
		200	88.0	10.7	1.3	95.1	4.1	0.8	88.7	9.7	1.6
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t})$, $h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t}$, $(\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1))$, $i = 1, ..., p$											
λ	σ_ξ	T									
0.936	0.424	50	80.7	16.8	2.5	91.6	7.3	1.1	83.1	14.0	2.9
		100	78.7	19.0	2.4	93.2	5.9	1.0	80.9	16.4	2.6
		200	78.0	19.9	2.1	93.2	6.2	0.6	79.9	17.7	2.4
0.951	0.314	50	83.5	14.4	2.1	92.9	6.0	1.1	86.3	11.4	2.3
		100	82.5	15.6	1.9	93.5	5.5	1.0	84.4	13.2	2.3
		200	81.4	16.9	1.7	93.4	6.1	0.6	82.8	15.2	2.0

TABLE 2

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =3, True Rank is 0.

			Q-based				Q^b-based				Q^s-based			
		$r=$	0	1	2	3	0	1	2	3	0	1	2	3
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$														
d_0	d_1	T												
0.0	0.0	50	93.0	6.1	0.7	0.1	94.9	4.2	0.6	0.3	95.1	4.0	0.7	0.2
		100	93.4	6.0	0.6	0.1	94.9	4.4	0.5	0.2	94.5	4.8	0.6	0.1
		200	93.9	5.5	0.6	0.0	94.7	4.6	0.5	0.1	94.8	4.5	0.5	0.2
0.5	0.0	50	89.3	9.7	0.9	0.1	93.4	5.6	0.7	0.3	92.2	7.0	0.6	0.2
		100	90.2	8.8	0.9	0.1	93.7	5.5	0.6	0.2	92.1	6.9	0.8	0.2
		200	91.7	7.5	0.7	0.1	94.7	4.7	0.5	0.1	92.8	6.3	0.6	0.3
0.3	0.65	50	87.4	11.2	1.1	0.3	92.8	5.9	1.1	0.2	90.4	8.1	1.1	0.4
		100	87.7	11.1	1.0	0.2	93.5	5.7	0.6	0.3	89.7	8.8	1.1	0.3
		200	89.3	9.8	0.8	0.1	94.8	4.6	0.4	0.2	90.5	8.4	0.9	0.2
0.2	0.79	50	88.8	10.0	0.9	0.2	92.9	5.9	0.9	0.3	91.7	7.2	0.8	0.3
		100	88.6	10.1	1.1	0.2	93.6	5.4	0.8	0.2	90.2	8.4	1.1	0.3
		200	87.8	11.0	1.0	0.2	94.6	4.6	0.6	0.2	89.0	9.5	1.1	0.4
0.05	0.94	50	92.4	6.6	0.8	0.2	94.5	4.5	0.6	0.4	94.8	4.3	0.7	0.2
		100	93.0	6.2	0.6	0.2	94.6	4.5	0.6	0.2	94.4	4.9	0.6	0.2
		200	92.8	6.6	0.4	0.1	94.9	4.5	0.5	0.2	93.5	5.9	0.4	0.2
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$														
d_0	d_1	T												
0.0	0.0	50	92.0	7.3	0.6	0.2	95.1	4.2	0.5	0.2	94.2	5.1	0.5	0.3
		100	93.7	5.7	0.4	0.1	95.3	4.0	0.5	0.2	95.1	4.2	0.4	0.2
		200	94.2	5.2	0.6	0.0	95.4	4.0	0.5	0.1	95.4	4.1	0.5	0.1
0.5	0.0	50	88.7	10.3	0.8	0.1	93.6	5.4	0.8	0.2	92.1	6.8	0.9	0.2
		100	91.6	7.7	0.5	0.2	94.8	4.4	0.7	0.2	93.2	6.0	0.7	0.2
		200	93.1	6.4	0.5	0.1	95.3	4.2	0.4	0.1	94.1	5.3	0.4	0.2
0.3	0.65	50	89.0	9.9	0.9	0.1	93.8	5.4	0.6	0.2	92.2	6.9	0.7	0.2
		100	90.8	8.3	0.7	0.2	94.5	4.6	0.6	0.3	92.3	6.7	0.8	0.3
		200	91.8	7.5	0.6	0.1	94.8	4.6	0.4	0.2	92.9	6.2	0.6	0.3
0.2	0.79	50	89.5	9.4	1.0	0.1	94.0	5.2	0.6	0.2	92.4	6.5	0.9	0.3
		100	91.1	7.9	0.8	0.2	94.6	4.6	0.6	0.3	92.7	6.2	0.7	0.3
		200	91.7	7.6	0.6	0.1	95.0	4.4	0.5	0.2	93.0	6.2	0.5	0.3
0.05	0.94	50	91.1	8.1	0.7	0.1	94.6	4.6	0.5	0.3	93.8	5.4	0.6	0.2
		100	92.9	6.5	0.4	0.2	95.0	4.2	0.5	0.3	94.1	5.1	0.4	0.3
		200	93.5	6.0	0.5	0.1	95.4	4.1	0.4	0.1	94.5	4.8	0.4	0.2
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}], v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$														
		50	86.2	12.1	1.4	0.3	92.1	6.5	1.0	0.4	89.5	8.9	1.2	0.4
		100	87.1	11.6	1.2	0.1	93.4	5.7	0.7	0.2	89.3	9.1	1.4	0.3
		200	88.5	10.4	1.0	0.1	94.1	5.4	0.3	0.2	89.9	8.7	1.0	0.4
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$														
		50	85.6	12.6	1.5	0.3	92.1	6.5	1.1	0.4	88.9	9.4	1.2	0.5
		100	85.0	13.3	1.5	0.2	93.1	5.9	0.8	0.2	86.9	11.3	1.4	0.4
		200	83.0	15.0	1.8	0.3	94.0	5.2	0.6	0.2	85.0	12.6	1.9	0.5
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$														
		50	86.9	11.6	1.3	0.2	92.7	6.1	0.9	0.3	89.9	8.5	1.0	0.6
		100	87.0	11.7	1.2	0.2	93.7	5.5	0.6	0.1	88.8	9.7	1.2	0.3
		200	85.9	12.8	1.2	0.1	94.3	5.1	0.5	0.1	87.5	10.7	1.5	0.3
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t},\ (\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$														
λ	σ_ξ													
0.936	0.424	50	75.5	20.7	3.4	0.4	90.9	7.5	1.3	0.3	80.9	15.9	2.6	0.6
		100	73.2	22.3	3.8	0.6	91.5	7.4	0.9	0.2	76.8	19.1	3.2	0.8
		200	72.7	23.6	3.3	0.4	92.4	6.7	0.7	0.2	75.4	21.4	2.5	0.7
0.951	0.314	50	80.0	17.2	2.4	0.4	91.8	6.7	1.2	0.3	83.7	13.6	2.1	0.6
		100	77.8	18.8	3.0	0.5	92.6	6.4	0.9	0.2	80.8	15.9	2.5	0.8
		200	77.2	20.0	2.4	0.4	93.3	6.1	0.5	0.1	79.5	17.6	2.3	0.6

TABLE 3

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =4, True Rank is 0.

			Q-based					Q^b-based					Q^s-based				
		$r=$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	91.3	7.7	0.9	0.1	0.0	95.7	3.7	0.5	0.1	0.1	95.2	4.1	0.5	0.1	0.0
		100	92.7	6.4	0.7	0.2	0.0	95.0	4.2	0.6	0.2	0.0	94.9	4.3	0.6	0.2	0.0
		200	93.5	5.9	0.5	0.1	0.0	95.1	4.2	0.5	0.1	0.1	95.1	4.2	0.5	0.1	0.0
0.5	0.0	50	86.4	12.0	1.3	0.2	0.0	93.7	5.4	0.6	0.2	0.2	91.5	7.4	0.8	0.2	0.1
		100	88.8	10.0	1.0	0.2	0.0	93.7	5.5	0.4	0.2	0.1	91.7	7.1	0.7	0.3	0.1
		200	91.3	8.0	0.6	0.1	0.0	94.8	4.7	0.4	0.1	0.0	93.3	5.9	0.6	0.1	0.0
0.3	0.65	50	85.2	13.1	1.4	0.2	0.0	93.6	5.4	0.7	0.2	0.1	90.6	8.0	1.0	0.3	0.2
		100	86.3	12.0	1.5	0.2	0.0	93.8	5.3	0.7	0.2	0.1	89.3	9.2	1.3	0.2	0.1
		200	87.9	10.9	1.0	0.2	0.0	94.4	4.9	0.6	0.1	0.0	90.0	8.8	1.0	0.2	0.1
0.2	0.79	50	86.2	12.1	1.4	0.2	0.0	94.1	4.9	0.6	0.3	0.1	91.8	6.9	0.8	0.3	0.1
		100	86.9	11.5	1.3	0.2	0.0	93.8	5.3	0.8	0.1	0.0	90.1	8.3	1.2	0.3	0.1
		200	87.2	11.4	1.2	0.1	0.1	94.5	4.8	0.6	0.1	0.0	89.3	9.3	1.2	0.1	0.1
0.05	0.94	50	90.7	8.2	1.0	0.1	0.1	95.4	3.9	0.4	0.2	0.1	94.7	4.5	0.5	0.2	0.1
		100	91.9	7.3	0.7	0.1	0.0	94.8	4.5	0.4	0.3	0.1	94.2	4.9	0.6	0.2	0.0
		200	92.8	6.5	0.6	0.1	0.0	95.0	4.4	0.5	0.1	0.0	94.5	4.8	0.6	0.1	0.0
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	90.7	8.2	0.9	0.2	0.0	95.6	3.6	0.6	0.1	0.1	94.4	4.9	0.5	0.2	0.0
		100	93.3	5.8	0.6	0.2	0.1	95.9	3.4	0.5	0.1	0.1	95.0	4.4	0.4	0.2	0.1
		200	93.7	5.6	0.6	0.0	0.0	95.2	4.2	0.6	0.0	0.0	95.1	4.2	0.7	0.1	0.0
0.5	0.0	50	87.3	11.2	1.1	0.3	0.0	94.4	4.8	0.6	0.2	0.0	91.8	7.2	0.7	0.2	0.1
		100	90.5	8.4	0.9	0.1	0.1	94.9	4.3	0.6	0.1	0.1	93.1	5.9	0.7	0.2	0.1
		200	91.9	7.4	0.6	0.1	0.0	95.1	4.4	0.4	0.1	0.1	93.6	5.8	0.4	0.2	0.1
0.3	0.65	50	87.4	10.9	1.4	0.2	0.0	94.1	5.1	0.6	0.1	0.1	92.1	6.8	0.8	0.2	0.1
		100	89.6	9.2	0.9	0.1	0.1	94.5	4.6	0.6	0.1	0.1	92.3	6.5	0.8	0.2	0.1
		200	90.5	8.6	0.7	0.1	0.0	94.9	4.5	0.5	0.1	0.1	92.6	6.6	0.6	0.1	0.1
0.2	0.79	50	88.3	10.1	1.3	0.2	0.1	94.8	4.5	0.5	0.1	0.1	92.7	6.3	0.7	0.2	0.1
		100	90.3	8.6	0.9	0.1	0.1	94.7	4.5	0.6	0.1	0.1	92.7	6.1	0.8	0.2	0.2
		200	91.3	7.8	0.6	0.2	0.0	94.9	4.4	0.5	0.1	0.1	92.6	6.5	0.7	0.1	0.1
0.05	0.94	50	90.1	8.8	0.9	0.2	0.0	95.4	3.8	0.6	0.1	0.1	94.1	5.1	0.6	0.2	0.0
		100	92.6	6.5	0.7	0.2	0.1	95.4	3.9	0.5	0.1	0.1	94.6	4.5	0.6	0.2	0.1
		200	92.8	6.5	0.6	0.0	0.0	94.9	4.5	0.5	0.1	0.0	94.2	5.0	0.7	0.0	0.0
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}], v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
		50	82.8	15.1	1.8	0.2	0.0	92.7	6.3	0.8	0.2	0.1	89.4	9.0	1.1	0.4	0.1
		100	85.4	12.8	1.6	0.2	0.0	93.7	5.3	0.7	0.2	0.1	88.9	9.4	1.2	0.3	0.1
		200	86.4	12.1	1.3	0.2	0.0	94.6	4.6	0.6	0.1	0.0	88.8	9.7	1.2	0.2	0.0
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2, v_{i,t} \sim$ i.i.d. $N(0,1), i = 1, ..., p$																	
		50	83.1	14.6	2.0	0.3	0.0	92.4	6.4	0.8	0.3	0.0	88.7	9.7	1.2	0.3	0.0
		100	83.4	14.1	2.1	0.3	0.1	93.1	5.9	0.8	0.1	0.1	86.9	10.9	1.8	0.4	0.1
		200	82.1	15.4	2.2	0.2	0.1	93.7	5.4	0.6	0.2	0.0	84.7	13.0	1.9	0.3	0.1
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2, v_{i,t} \sim$ i.i.d. $N(0,1), i = 1, ..., p$																	
		50	84.5	13.6	1.7	0.2	0.0	93.4	5.6	0.8	0.2	0.0	89.9	8.7	1.1	0.2	0.1
		100	85.1	13.1	1.4	0.2	0.1	93.8	5.3	0.7	0.1	0.1	88.2	10.1	1.3	0.2	0.1
		200	84.0	14.1	1.7	0.1	0.0	94.4	5.0	0.4	0.2	0.0	86.4	11.8	1.6	0.2	0.0
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t},\ (\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$																	
λ	σ_ξ	T															
0.936	0.424	50	70.7	23.9	4.4	0.8	0.2	90.3	8.2	1.1	0.2	0.2	79.2	16.8	2.9	0.7	0.4
		100	67.8	26.0	5.2	0.8	0.2	91.3	7.5	0.9	0.2	0.1	73.7	21.3	4.0	0.7	0.3
		200	67.3	26.8	5.0	0.8	0.1	92.2	6.9	0.8	0.0	0.0	71.9	23.3	3.9	0.6	0.3
0.951	0.314	50	76.0	20.0	3.4	0.5	0.1	91.6	7.0	1.0	0.2	0.2	83.5	13.7	2.2	0.4	0.2
		100	74.6	20.8	3.9	0.4	0.2	92.1	6.7	1.0	0.1	0.1	79.2	16.9	3.1	0.5	0.3
		200	74.1	21.1	4.0	0.6	0.1	93.4	5.5	0.8	0.2	0.1	77.8	18.2	3.2	0.6	0.2

TABLE 4

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =5, True Rank is 0.

			Q-based					Q^b-based					Q^s-based				
		$r=$	0	1	2	3	4,5	0	1	2	3	4,5	0	1	2	3	4,5
Model A: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}, h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}, v_{i,t}\sim i.i.d.\ N(0,1),\ i=1,...,p$																	
d_0	d_1	T															
0.0	0.0	50	87.9	10.6	1.2	0.3	0.1	96.3	3.1	0.4	0.2	0.0	95.2	4.1	0.6	0.1	0.0
		100	91.9	7.3	0.6	0.1	0.0	95.6	4.0	0.4	0.0	0.0	95.3	4.3	0.3	0.1	0.0
		200	93.1	6.1	0.7	0.1	0.0	95.6	3.7	0.6	0.1	0.0	95.3	4.0	0.5	0.1	0.0
0.5	0.0	50	82.4	15.3	2.0	0.3	0.0	93.8	5.4	0.6	0.2	0.0	90.9	8.3	0.8	0.1	0.0
		100	87.4	11.0	1.4	0.1	0.0	94.6	4.7	0.6	0.1	0.1	91.8	7.1	0.8	0.2	0.1
		200	90.4	8.8	0.8	0.1	0.0	95.7	3.9	0.4	0.0	0.0	93.2	6.0	0.6	0.1	0.0
0.3	0.65	50	81.9	15.3	2.5	0.2	0.1	93.8	5.2	0.9	0.1	0.1	90.5	8.1	1.2	0.1	0.0
		100	85.2	12.8	1.5	0.4	0.1	93.7	5.4	0.6	0.1	0.1	90.0	8.7	0.9	0.3	0.1
		200	86.1	12.3	1.4	0.2	0.0	94.3	5.1	0.6	0.1	0.0	89.1	9.5	1.2	0.2	0.1
0.2	0.79	50	83.8	13.8	2.1	0.3	0.1	94.5	4.5	0.8	0.2	0.0	92.3	6.4	1.0	0.3	0.1
		100	86.0	12.1	1.5	0.3	0.1	94.5	4.8	0.5	0.1	0.0	90.8	7.9	1.0	0.2	0.1
		200	86.5	11.8	1.5	0.2	0.0	94.4	4.8	0.7	0.1	0.0	89.4	9.0	1.3	0.3	0.0
0.05	0.94	50	87.7	10.8	1.2	0.3	0.1	95.8	3.5	0.5	0.2	0.0	95.1	4.1	0.6	0.2	0.0
		100	91.2	7.8	0.8	0.1	0.1	95.6	3.8	0.5	0.1	0.0	95.1	4.3	0.6	0.1	0.0
		200	92.1	7.1	0.7	0.1	0.0	95.1	4.3	0.5	0.1	0.0	94.5	4.9	0.6	0.1	0.0
Model B: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}, h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}, v_{i,t}\sim i.i.d.\ t_5,\ i=1,...,p$																	
d_0	d_1	T															
0.0	0.0	50	87.2	11.4	1.1	0.2	0.1	96.4	3.2	0.2	0.1	0.1	95.0	4.5	0.3	0.1	0.1
		100	91.8	7.3	0.8	0.1	0.0	95.6	3.9	0.3	0.1	0.0	95.2	4.3	0.4	0.0	0.1
		200	93.5	5.9	0.5	0.0	0.0	96.2	3.4	0.3	0.0	0.0	95.6	4.0	0.3	0.0	0.1
0.5	0.0	50	84.1	13.9	1.7	0.2	0.1	95.2	4.0	0.6	0.1	0.1	92.6	6.6	0.7	0.1	0.1
		100	88.0	10.6	1.2	0.1	0.0	94.9	4.4	0.5	0.1	0.0	92.8	6.4	0.7	0.1	0.0
		200	91.4	7.8	0.7	0.1	0.0	95.7	3.8	0.4	0.1	0.0	93.9	5.4	0.6	0.1	0.0
0.3	0.65	50	84.2	13.8	1.7	0.2	0.1	95.1	4.0	0.6	0.1	0.1	92.8	6.0	1.0	0.1	0.1
		100	87.6	11.1	1.1	0.1	0.1	94.4	5.0	0.4	0.1	0.0	92.5	6.6	0.7	0.1	0.1
		200	89.8	9.3	1.0	0.0	0.0	95.3	4.2	0.4	0.1	0.0	92.8	6.4	0.7	0.1	0.0
0.2	0.79	50	85.5	12.5	1.6	0.3	0.1	95.3	3.9	0.6	0.1	0.1	93.6	5.3	0.9	0.1	0.1
		100	88.8	9.9	1.2	0.1	0.0	94.6	4.8	0.4	0.1	0.1	92.8	6.3	0.7	0.1	0.1
		200	90.3	8.7	1.0	0.1	0.0	95.5	4.1	0.4	0.1	0.0	93.4	6.0	0.6	0.1	0.0
0.05	0.94	50	86.9	11.5	1.3	0.2	0.1	96.1	3.4	0.3	0.1	0.1	94.8	4.7	0.4	0.1	0.0
		100	91.1	8.0	0.9	0.1	0.0	95.5	3.9	0.5	0.1	0.0	94.6	4.7	0.6	0.1	0.0
		200	92.8	6.6	0.6	0.0	0.0	96.0	3.7	0.3	0.1	0.0	95.2	4.3	0.4	0.1	0.0
Model C: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}, \ln(h_{i,t})=-0.23+0.9\ln(h_{i,t-1})+0.25[\lvert v_{i,t-1}^2\rvert-0.3v_{i,t-1}], v_{i,t}\sim i.i.d.\ N(0,1),\ i=1,...,p$																	
		50	80.0	16.9	2.7	0.3	0.1	92.9	5.9	0.9	0.1	0.1	89.9	8.7	1.1	0.2	0.1
		100	83.2	14.7	1.8	0.3	0.1	93.0	5.9	0.9	0.2	0.0	88.2	10.1	1.2	0.4	0.1
		200	85.3	13.0	1.6	0.1	0.0	94.3	5.0	0.6	0.1	0.0	88.8	9.9	1.2	0.2	0.0
Model D: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t},\ h_{i,t}=0.0216+0.6896h_{i,t-1}+0.3174[\varepsilon_{i,t-1}-0.1108]^2,\ v_{i,t}\sim$ i.i.d. $N(0,1),\ i=1,...,p$																	
		50	80.2	16.5	2.7	0.5	0.1	93.4	5.4	0.8	0.3	0.1	89.7	8.3	1.6	0.4	0.1
		100	81.3	15.9	2.4	0.4	0.1	93.5	5.6	0.7	0.2	0.0	86.8	11.1	1.6	0.4	0.1
		200	79.8	17.1	2.8	0.3	0.1	93.6	5.3	0.9	0.1	0.1	83.9	13.3	2.2	0.5	0.1
Model E: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t},\ h_{i,t}=0.005+0.7h_{i,t-1}+0.28[\lvert\varepsilon_{i,t-1}\rvert-0.23\varepsilon_{i,t-1}]^2,\ v_{i,t}\sim$ i.i.d. $N(0,1),\ i=1,...,p$																	
		50	81.8	15.2	2.6	0.4	0.1	93.9	4.9	0.9	0.2	0.1	90.5	7.8	1.4	0.2	0.1
		100	83.9	13.9	1.9	0.3	0.1	94.2	5.1	0.6	0.1	0.1	88.6	9.8	1.3	0.2	0.1
		200	83.1	14.7	1.8	0.3	0.0	94.6	4.6	0.5	0.1	0.0	86.2	11.7	1.6	0.3	0.1
Model F: $\varepsilon_{i,t}=v_{i,t}\exp(h_{i,t}),\ h_{i,t}=\lambda h_{i,t-1}+0.5\xi_{i,t},\ (\xi_{i,t},v_{i,t})\sim$ i.i.d. $N(0,\mathrm{diag}(\sigma_\xi^2,1)),\ i=1,...,p$																	
λ	σ_ξ	T															
0.936	0.424	50	65.0	26.4	6.9	1.3	0.3	89.0	8.9	1.6	0.4	0.1	77.8	17.0	3.9	1.0	0.3
		100	64.6	27.8	6.2	1.2	0.2	90.5	7.9	1.2	0.3	0.1	73.0	21.4	4.6	0.8	0.2
		200	62.9	28.7	7.0	1.3	0.1	92.1	6.6	1.1	0.2	0.0	69.2	23.9	5.6	1.1	0.2
0.951	0.314	50	71.9	22.0	5.0	1.0	0.2	91.0	7.4	1.2	0.2	0.1	82.7	13.8	2.7	0.6	0.2
		100	72.0	22.5	4.5	0.9	0.2	91.3	7.3	1.2	0.2	0.0	78.5	17.6	3.0	0.8	0.2
		200	69.5	24.8	4.8	0.8	0.1	92.3	6.7	0.8	0.2	0.0	75.1	20.4	3.5	0.9	0.1

TABLE 5

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =2, True Rank is 1.

			Q-based			Q^b-based			Q^s-based		
			$r=0$	$r=1$	$r=2$	$r=0$	$r=1$	$r=2$	$r=0$	$r=1$	$r=2$
Model A: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$											
d_0	d_1	T									
0.0	0.0	50	9.4	85.4	5.2	14.8	79.5	5.7	12.0	83.1	4.8
		100	0.0	94.2	5.8	0.0	94.4	5.6	0.0	94.5	5.5
		200	0.0	94.5	5.5	0.0	94.4	5.6	0.0	95.0	5.0
0.5	0.0	50	9.8	83.9	6.3	18.0	76.4	5.6	12.5	81.7	5.8
		100	0.0	93.6	6.4	0.3	93.8	6.0	0.0	94.0	6.0
		200	0.0	94.9	5.1	0.0	95.1	4.9	0.0	95.0	5.0
0.3	0.65	50	12.5	80.7	6.8	21.2	73.1	5.8	14.7	79.0	6.2
		100	0.2	92.0	7.7	1.4	92.8	5.8	0.3	92.3	7.4
		200	0.0	92.5	7.5	0.1	94.4	5.5	0.0	92.6	7.4
0.2	0.79	50	14.3	79.3	6.5	22.3	72.0	5.7	17.0	76.9	6.1
		100	0.3	91.7	8.0	1.8	92.4	5.9	0.3	92.2	7.5
		200	0.0	92.1	7.9	0.1	94.6	5.4	0.0	92.2	7.8
0.05	0.94	50	11.7	83.1	5.2	16.6	77.9	5.5	14.2	80.8	5.0
		100	0.0	93.9	6.1	0.2	94.2	5.6	0.0	94.0	5.9
		200	0.0	94.2	5.8	0.0	94.3	5.7	0.0	94.3	5.7
Model B: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim$ i.i.d. t_5, $i=1,...,p$											
d_0	d_1	T									
0.0	0.0	50	10.2	84.6	5.1	16.1	78.1	5.9	13.5	81.9	4.6
		100	0.0	94.4	5.6	0.1	94.2	5.6	0.0	94.8	5.1
		200	0.0	95.1	4.9	0.0	95.4	4.6	0.0	95.7	4.3
0.5	0.0	50	10.4	83.4	6.2	17.9	75.9	6.2	13.7	80.8	5.6
		100	0.0	93.7	6.3	0.3	93.6	6.2	0.1	94.1	5.8
		200	0.0	94.5	5.5	0.0	95.2	4.8	0.0	95.1	4.9
0.3	0.65	50	11.3	82.2	6.5	19.1	74.6	6.3	14.7	79.1	6.2
		100	0.1	93.3	6.6	0.6	93.5	5.9	0.1	93.7	6.1
		200	0.0	94.1	5.9	0.0	95.1	4.9	0.0	94.6	5.4
0.2	0.79	50	12.0	81.7	6.4	19.0	74.8	6.2	15.2	78.9	5.9
		100	0.1	93.4	6.5	0.6	93.4	6.0	0.2	93.9	6.0
		200	0.0	94.2	5.8	0.0	95.5	4.5	0.0	94.4	5.6
0.05	0.94	50	11.4	82.9	5.7	17.3	76.7	6.0	14.4	80.7	5.0
		100	0.1	93.9	6.0	0.3	94.2	5.5	0.1	94.3	5.6
		200	0.0	94.5	5.5	0.0	95.2	4.8	0.0	95.0	5.0
Model C: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $\ln(h_{i,t})=-0.23+0.9\ln(h_{i,t-1})+0.25[\lvert v_{i,t-1}^2\rvert-0.3v_{i,t-1}]$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$											
		50	12.2	80.8	7.0	21.7	72.5	5.8	14.8	78.6	6.5
		100	0.2	92.1	7.7	1.4	92.7	5.9	0.3	92.5	7.2
		200	0.0	92.4	7.6	0.1	94.3	5.6	0.0	92.8	7.2
Model D: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.0216+0.6896h_{i,t-1}+0.3174[\varepsilon_{i,t-1}-0.1108]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$											
		50	14.7	77.5	7.8	24.5	69.7	5.9	17.3	75.5	7.2
		100	0.6	89.7	9.8	3.0	90.5	6.5	0.7	90.1	9.2
		200	0.0	89.5	10.5	0.4	93.7	5.9	0.0	89.7	10.2
Model E: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.005+0.7h_{i,t-1}+0.28[\lvert\varepsilon_{i,t-1}\rvert-0.23\varepsilon_{i,t-1}]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$											
		50	13.5	79.0	7.5	22.7	71.3	6.0	16.2	77.0	6.8
		100	0.5	90.3	9.2	2.5	91.3	6.2	0.5	91.0	8.5
		200	0.0	90.7	9.3	0.2	94.0	5.8	0.0	90.9	9.1
Model F: $\varepsilon_{i,t}=v_{i,t}\exp(h_{i,t})$, $h_{i,t}=\lambda h_{i,t-1}+0.5\xi_{i,t}$, $(\xi_{i,t},v_{i,t})\sim$ i.i.d. $N(0,\mathrm{diag}(\sigma_\xi^2,1))$, $i=1,...,p$											
λ	σ_ξ	T									
0.936	0.424	50	16.0	73.5	10.5	29.5	64.5	6.0	19.3	70.8	9.9
		100	1.3	86.3	12.4	8.7	85.0	6.3	1.9	86.5	11.6
		200	0.0	88.0	12.0	1.2	92.7	6.0	0.0	88.5	11.5
0.951	0.314	50	15.9	74.9	9.2	27.8	66.2	6.0	18.9	72.3	8.8
		100	0.9	87.8	11.3	6.3	87.0	6.7	1.3	88.0	10.7
		200	0.0	89.1	10.9	0.7	93.8	5.5	0.0	89.6	10.4

TABLE 6

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p =3, True Rank is 1.

			Q-based				Q^b-based				Q^s-based			
		$r=$	0	1	2	3	0	1	2	3	0	1	2	3
Model A: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$														
d_0	d_1	T												
0.0	0.0	50	28.5	66.3	4.8	0.4	40.3	55.0	3.8	0.9	36.7	59.3	3.4	0.6
		100	0.2	93.9	5.5	0.5	0.4	94.3	4.5	0.7	0.3	94.7	4.4	0.6
		200	0.0	94.4	5.2	0.4	0.0	95.0	4.5	0.5	0.0	95.1	4.4	0.5
0.5	0.0	50	25.5	66.9	6.8	0.8	39.8	54.9	4.3	1.0	32.4	61.7	5.0	0.9
		100	0.8	91.3	7.3	0.6	2.1	92.3	4.7	0.8	1.1	92.4	5.7	0.9
		200	0.0	92.7	6.7	0.6	0.0	94.6	4.6	0.8	0.0	93.5	5.7	0.8
0.3	0.65	50	25.9	66.4	6.8	0.8	39.8	54.9	4.3	1.1	31.6	62.5	4.9	1.0
		100	1.8	88.0	9.1	1.1	5.1	89.0	5.0	1.0	2.4	89.4	6.9	1.3
		200	0.0	90.6	8.6	0.8	0.2	94.7	4.3	0.8	0.0	91.3	7.6	1.2
0.2	0.79	50	27.3	64.9	7.0	0.7	38.8	55.7	4.5	0.9	32.6	61.4	5.2	0.8
		100	2.5	87.4	9.0	1.1	6.1	88.3	4.7	0.9	3.1	89.0	6.6	1.3
		200	0.0	89.6	9.4	1.0	0.2	93.8	5.1	0.9	0.0	90.8	7.9	1.3
0.05	0.94	50	27.7	66.5	5.3	0.5	37.6	57.9	3.5	1.0	34.2	61.4	3.7	0.7
		100	0.8	92.3	6.3	0.6	1.9	93.1	4.3	0.8	1.6	92.9	4.8	0.8
		200	0.0	93.2	6.4	0.4	0.0	94.9	4.4	0.7	0.0	94.3	5.1	0.6
Model B: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim$ i.i.d. t_5, $i=1,...,p$														
d_0	d_1	T												
0.0	0.0	50	28.6	64.9	5.9	0.6	40.9	53.8	4.1	1.2	36.8	58.5	3.8	0.9
		100	0.5	93.1	5.7	0.7	1.1	93.7	4.2	1.0	0.8	93.7	4.7	0.8
		200	0.0	94.4	5.1	0.5	0.0	95.4	4.0	0.7	0.0	95.4	4.1	0.5
0.5	0.0	50	26.6	65.6	7.2	0.5	40.7	53.9	4.3	1.0	34.7	59.3	5.1	0.9
		100	0.5	92.6	6.2	0.7	2.0	92.8	4.2	1.1	1.0	93.0	5.0	0.9
		200	0.0	93.5	6.0	0.5	0.0	94.8	4.7	0.4	0.0	94.5	5.0	0.5
0.3	0.65	50	27.3	64.8	7.1	0.7	41.0	53.5	4.4	1.1	34.3	59.4	5.2	1.2
		100	0.9	91.5	7.0	0.7	2.7	91.9	4.4	1.0	1.2	92.3	5.6	0.9
		200	0.0	92.8	6.6	0.6	0.0	95.0	4.3	0.7	0.0	93.9	5.3	0.8
0.2	0.79	50	27.9	64.6	6.8	0.7	40.3	54.4	4.4	1.0	34.9	59.2	4.8	1.1
		100	1.0	91.3	7.0	0.7	2.8	91.9	4.4	0.9	1.3	92.3	5.6	0.8
		200	0.0	92.6	6.9	0.5	0.0	95.1	4.2	0.6	0.0	93.7	5.7	0.6
0.05	0.94	50	28.5	64.8	6.1	0.6	40.5	54.5	4.0	1.0	36.1	58.9	4.0	0.9
		100	0.7	92.4	6.3	0.6	1.7	92.9	4.3	1.2	1.0	93.1	5.0	0.9
		200	0.0	93.8	5.7	0.5	0.0	95.2	4.2	0.6	0.0	95.0	4.4	0.6
Model C: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $\ln(h_{i,t})=-0.23+0.9\ln(h_{i,t-1})+0.25[\lvert v_{i,t-1}^2\rvert-0.3v_{i,t-1}]$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$														
		50	25.0	66.8	7.2	1.0	39.8	54.5	4.5	1.1	31.3	62.2	5.4	1.1
		100	1.7	88.0	9.3	1.0	5.4	88.4	5.3	1.0	2.3	88.8	7.7	1.2
		200	0.0	90.3	9.0	0.7	0.3	93.8	5.1	0.8	0.0	91.4	7.5	1.1
Model D: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.0216+0.6896h_{i,t-1}+0.3174[\varepsilon_{i,t-1}-0.1108]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$														
		50	26.6	64.1	8.4	1.0	40.1	54.4	4.6	0.8	31.6	61.0	6.3	1.1
		100	3.2	83.6	11.6	1.5	8.9	84.2	5.6	1.3	3.9	84.8	9.4	1.9
		200	0.0	85.9	12.5	1.6	0.7	93.0	5.1	1.1	0.0	87.3	10.8	1.9
Model E: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.005+0.7h_{i,t-1}+0.28[\lvert\varepsilon_{i,t-1}\rvert-0.23\varepsilon_{i,t-1}]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$														
		50	27.0	64.1	7.9	1.0	40.6	53.9	4.5	1.0	31.9	61.2	5.6	1.3
		100	2.8	85.5	10.4	1.3	7.9	85.7	5.5	1.0	3.4	86.6	8.2	1.7
		200	0.0	87.6	11.1	1.3	0.5	93.3	5.5	0.6	0.0	88.5	10.0	1.5
Model F: $\varepsilon_{i,t}=v_{i,t}\exp(h_{i,t})$, $h_{i,t}=\lambda h_{i,t-1}+0.5\xi_{i,t}$, $(\xi_{i,t},v_{i,t})\sim$ i.i.d. $N(0,\mathrm{diag}(\sigma_\xi^2,1))$, $i=1,...,p$														
0.936	0.424	50	23.4	61.5	13.1	2.0	42.0	51.4	5.3	1.3	28.7	59.7	9.6	2.1
		100	3.7	76.7	17.0	2.6	18.2	74.5	6.2	1.1	5.2	77.9	14.1	2.8
		200	0.1	81.0	16.9	2.0	2.3	91.1	5.8	0.8	0.1	82.9	14.6	2.4
0.951	0.314	50	24.9	62.5	11.0	1.7	41.1	52.4	5.1	1.4	29.9	59.7	8.4	2.1
		100	3.5	79.6	14.7	2.2	14.7	78.3	5.6	1.4	4.5	81.2	11.8	2.6
		200	0.1	83.8	14.7	1.5	1.5	92.4	5.2	0.8	0.1	85.6	12.3	2.0

TABLE 7

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p = 4, True Rank is 1.

			Q-based					Q^b-based					Q^s-based				
		$r =$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	42.5	51.2	5.5	0.7	0.1	60.3	35.7	3.1	0.6	0.3	56.1	40.2	3.0	0.4	0.2
		100	2.2	90.9	6.3	0.6	0.0	4.6	90.3	4.3	0.6	0.3	3.6	91.4	4.3	0.5	0.2
		200	0.0	94.7	4.8	0.4	0.0	0.0	95.5	3.9	0.5	0.1	0.0	95.5	4.0	0.4	0.1
0.5	0.0	50	36.6	54.3	8.0	0.8	0.3	58.0	37.1	4.1	0.5	0.3	48.5	45.9	4.8	0.5	0.4
		100	2.9	88.6	7.5	0.9	0.1	7.4	87.4	4.5	0.6	0.2	4.3	89.4	5.5	0.6	0.1
		200	0.0	92.2	7.2	0.5	0.1	0.0	94.8	4.6	0.4	0.1	0.0	93.9	5.4	0.5	0.1
0.3	0.65	50	34.5	55.9	8.4	0.8	0.3	54.6	40.4	4.0	0.5	0.4	45.5	48.3	5.3	0.6	0.3
		100	4.9	83.9	9.7	1.3	0.2	12.6	81.1	5.3	0.6	0.3	7.5	84.0	7.2	0.9	0.5
		200	0.0	89.5	9.2	1.1	0.2	0.4	94.6	4.3	0.5	0.2	0.0	91.2	7.5	1.0	0.2
0.2	0.79	50	35.7	55.5	7.5	1.0	0.3	51.8	43.4	3.9	0.5	0.3	45.2	49.7	4.2	0.6	0.3
		100	6.5	82.9	9.2	1.3	0.2	14.1	80.4	4.6	0.6	0.3	8.9	82.8	6.9	1.1	0.4
		200	0.0	88.8	9.9	1.0	0.2	0.4	94.3	4.5	0.6	0.3	0.0	90.7	7.9	1.0	0.4
0.05	0.94	50	39.5	54.3	5.2	0.8	0.1	56.0	39.9	3.3	0.5	0.3	51.0	45.4	2.9	0.4	0.3
		100	4.2	88.7	6.1	0.9	0.1	7.5	87.2	4.3	0.8	0.2	6.1	88.7	4.2	0.7	0.3
		200	0.0	93.4	5.9	0.7	0.0	0.0	95.4	4.0	0.5	0.2	0.0	94.6	4.6	0.5	0.2
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}$, $v_{i,t} \sim$ i.i.d. t_5, $i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	43.4	49.9	5.7	0.8	0.1	61.9	34.1	3.1	0.7	0.2	57.0	39.2	3.0	0.8	0.1
		100	2.7	90.6	5.9	0.6	0.1	5.9	89.5	3.9	0.5	0.3	4.3	90.8	4.1	0.5	0.2
		200	0.0	93.6	5.9	0.4	0.1	0.0	95.4	4.1	0.4	0.1	0.0	95.2	4.1	0.5	0.2
0.5	0.0	50	39.6	52.6	6.8	0.9	0.1	58.8	36.9	3.5	0.8	0.2	51.6	43.4	4.1	0.7	0.2
		100	3.0	88.3	7.9	0.7	0.1	8.1	86.5	4.7	0.5	0.3	4.9	88.7	5.6	0.6	0.2
		200	0.0	92.6	6.6	0.7	0.0	0.0	95.2	4.2	0.4	0.1	0.0	94.1	5.1	0.6	0.1
0.3	0.65	50	39.6	52.2	7.2	0.8	0.1	58.4	37.2	3.4	0.7	0.3	51.5	43.4	4.1	0.8	0.2
		100	3.7	87.2	7.9	0.9	0.2	9.7	84.9	4.5	0.6	0.3	6.0	87.3	5.8	0.7	0.3
		200	0.0	91.7	7.6	0.7	0.1	0.0	95.0	4.3	0.5	0.1	0.0	93.2	5.9	0.8	0.2
0.2	0.79	50	40.4	52.1	6.5	0.9	0.2	58.6	37.2	3.2	0.8	0.3	52.3	42.8	4.0	0.7	0.2
		100	4.4	86.8	7.5	1.0	0.2	10.0	84.7	4.4	0.7	0.3	6.6	86.9	5.4	0.7	0.3
		200	0.0	91.6	7.7	0.6	0.1	0.1	95.0	4.3	0.5	0.1	0.0	93.1	6.3	0.5	0.1
0.05	0.94	50	42.3	51.3	5.4	0.7	0.2	60.1	36.1	2.9	0.6	0.3	54.2	41.8	3.2	0.5	0.3
		100	3.9	89.1	6.2	0.7	0.1	7.9	87.2	4.0	0.6	0.3	5.7	88.9	4.5	0.5	0.3
		200	0.0	93.1	6.3	0.4	0.2	0.0	95.0	4.3	0.5	0.1	0.0	94.4	4.9	0.4	0.2
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $\ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}]$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$																	
		50	33.9	55.7	9.0	1.2	0.2	54.2	40.6	4.1	0.7	0.3	44.4	48.6	5.8	0.8	0.4
		100	4.9	83.3	10.5	1.3	0.1	13.7	80.6	4.7	0.6	0.4	7.1	83.9	7.6	1.0	0.4
		200	0.0	88.7	10.1	1.0	0.2	0.5	93.7	5.0	0.6	0.3	0.0	90.4	8.4	1.0	0.3
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$																	
		50	33.3	55.8	9.3	1.4	0.3	51.6	43.1	4.2	0.9	0.3	42.4	50.4	5.9	1.1	0.3
		100	7.4	78.6	12.0	1.7	0.3	18.0	75.8	5.0	0.9	0.4	9.7	79.0	9.3	1.3	0.6
		200	0.1	84.4	13.5	1.8	0.2	1.6	92.7	4.7	0.7	0.3	0.1	86.5	11.1	1.8	0.5
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}$, $h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2$, $v_{i,t} \sim$ i.i.d. $N(0,1)$, $i = 1, ..., p$																	
		50	34.5	54.9	9.2	1.1	0.3	52.4	42.6	3.9	0.7	0.3	43.4	50.3	5.3	0.7	0.3
		100	6.7	80.7	10.7	1.5	0.4	16.9	77.2	4.7	0.9	0.3	9.4	80.4	8.2	1.3	0.6
		200	0.1	86.0	12.3	1.4	0.3	0.9	93.2	5.0	0.5	0.3	0.1	88.0	10.2	1.3	0.4
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t})$, $h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t}$, $(\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1))$, $i = 1, ..., p$																	
λ	σ_ξ	T															
0.936	0.424	50	27.9	52.7	16.1	3.0	0.4	49.6	42.9	6.3	0.9	0.3	36.0	50.2	10.9	2.2	0.6
		100	7.9	68.1	20.3	3.2	0.4	26.7	65.4	6.4	1.2	0.3	10.9	69.5	16.2	2.7	0.7
		200	0.2	74.2	22.0	2.9	0.6	4.9	87.8	6.5	0.6	0.2	0.4	77.7	18.5	2.6	0.8
0.951	0.314	50	30.0	54.7	12.7	2.2	0.4	49.9	43.7	5.3	0.8	0.3	38.5	51.0	8.4	1.6	0.6
		100	7.8	72.4	17.1	2.4	0.4	23.9	68.7	6.2	0.9	0.3	10.5	73.0	14.0	1.7	0.8
		200	0.2	78.9	18.1	2.5	0.4	3.0	90.5	5.8	0.6	0.1	0.2	81.9	15.2	2.0	0.7

TABLE 8

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
p = 5, True Rank is 1.

			Q-based					Q^b-based					Q^s-based				
		$r=$	0	1	2	3	4, 5	0	1	2	3	4, 5	0	1	2	3	4, 5
Model A: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim i.i.d.\ N(0,1)$, $i=1,...,p$																	
d_0	d_1	T															
0.0	0.0	50	51.5	42.4	5.3	0.6	0.1	75.4	22.1	2.1	0.2	0.2	71.0	26.3	2.3	0.3	0.1
		100	7.2	85.5	6.6	0.6	0.1	15.5	80.0	3.8	0.5	0.2	12.9	82.5	4.0	0.5	0.1
		200	0.0	93.2	6.0	0.7	0.1	0.0	95.4	4.0	0.4	0.2	0.0	95.4	4.0	0.5	0.1
0.5	0.0	50	44.7	46.3	7.9	0.9	0.2	71.4	25.0	3.1	0.3	0.2	63.1	32.4	3.8	0.4	0.3
		100	7.7	81.7	9.4	1.0	0.2	19.9	74.3	5.0	0.6	0.2	12.8	79.8	6.5	0.7	0.2
		200	0.0	91.2	8.0	0.7	0.1	0.0	95.2	4.3	0.3	0.1	0.0	93.5	5.8	0.5	0.1
0.3	0.65	50	41.5	48.6	8.7	1.1	0.1	65.6	30.4	3.4	0.5	0.1	57.4	37.7	4.2	0.6	0.1
		100	10.3	77.5	10.5	1.6	0.2	24.6	69.9	4.6	0.7	0.2	15.5	75.8	7.3	1.1	0.3
		200	0.0	87.2	11.5	1.1	0.2	1.0	93.6	4.7	0.6	0.1	0.1	90.1	8.5	1.1	0.2
0.2	0.79	50	41.8	48.5	8.4	1.0	0.3	64.0	32.0	3.3	0.5	0.2	56.2	39.3	3.7	0.5	0.2
		100	12.2	75.7	10.5	1.3	0.3	25.4	69.5	4.2	0.7	0.2	17.6	74.4	6.8	1.0	0.3
		200	0.2	87.2	11.1	1.4	0.2	1.3	93.3	4.8	0.4	0.2	0.3	89.5	8.7	1.2	0.3
0.05	0.94	50	47.6	45.8	5.6	0.7	0.2	70.6	26.7	2.1	0.4	0.2	64.9	32.3	2.4	0.3	0.1
		100	10.4	81.7	7.0	0.7	0.1	19.3	75.9	4.0	0.6	0.1	16.1	79.1	4.2	0.5	0.1
		200	0.0	92.6	6.7	0.6	0.1	0.1	95.2	4.2	0.4	0.1	0.0	94.6	4.8	0.4	0.2
Model B: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=\omega+d_0\varepsilon_{i,t-1}^2+d_1h_{i,t-1}$, $v_{i,t}\sim i.i.d.\ t_5$, $i=1,...,p$																	
d_0	d_1	T															
0.0	0.0	50	49.8	42.9	6.4	0.8	0.1	74.6	22.7	2.3	0.3	0.1	69.1	27.7	2.8	0.3	0.1
		100	7.8	85.4	6.1	0.6	0.1	16.6	79.6	3.5	0.2	0.1	13.5	82.3	3.8	0.4	0.1
		200	0.0	93.4	6.0	0.6	0.1	0.0	95.9	3.8	0.3	0.1	0.0	95.3	4.2	0.4	0.1
0.5	0.0	50	43.6	46.5	8.9	0.7	0.2	71.1	25.4	3.0	0.3	0.2	63.1	32.6	3.9	0.3	0.2
		100	8.1	82.7	8.2	0.9	0.2	19.5	75.5	4.4	0.5	0.2	13.4	80.2	5.7	0.5	0.2
		200	0.0	92.0	7.4	0.5	0.1	0.2	95.1	4.3	0.3	0.1	0.0	94.4	5.1	0.3	0.2
0.3	0.65	50	43.1	47.2	8.8	0.8	0.1	70.6	26.0	2.8	0.3	0.2	61.8	33.6	4.2	0.4	0.1
		100	8.7	81.3	9.0	0.9	0.2	20.0	75.1	4.4	0.4	0.1	14.1	79.5	5.7	0.6	0.2
		200	0.0	91.0	8.1	0.7	0.2	0.3	94.9	4.2	0.5	0.1	0.0	93.3	5.8	0.7	0.1
0.2	0.79	50	44.0	47.0	8.0	0.8	0.1	70.0	26.8	2.7	0.4	0.1	62.4	33.4	3.7	0.4	0.1
		100	8.8	81.9	8.2	1.0	0.1	20.1	75.5	3.9	0.3	0.1	14.8	79.2	5.3	0.6	0.2
		200	0.0	91.0	8.2	0.7	0.1	0.2	95.0	4.2	0.5	0.1	0.0	93.4	5.7	0.6	0.2
0.05	0.94	50	47.8	44.5	6.9	0.7	0.1	72.6	24.4	2.5	0.3	0.2	66.3	30.3	2.8	0.4	0.1
		100	9.0	83.5	6.8	0.5	0.1	17.9	78.2	3.4	0.3	0.1	14.9	80.4	4.2	0.4	0.1
		200	0.0	92.7	6.6	0.7	0.1	0.1	95.6	3.9	0.4	0.0	0.0	94.8	4.5	0.5	0.1
Model C: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $\ln(h_{i,t})=-0.23+0.9\ln(h_{i,t-1})+0.25[\lvert v_{i,t-1}^2\rvert-0.3v_{i,t-1}]$, $v_{i,t}\sim i.i.d.\ N(0,1)$, $i=1,...,p$																	
		50	39.4	49.6	9.6	1.2	0.2	65.0	31.0	3.3	0.5	0.2	55.5	39.4	4.3	0.6	0.2
		100	9.2	76.9	12.1	1.7	0.1	23.9	69.6	5.6	0.8	0.2	14.4	76.0	8.3	1.1	0.3
		200	0.1	86.6	11.9	1.2	0.2	0.8	93.2	5.3	0.6	0.2	0.1	89.4	9.2	0.9	0.4
Model D: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.0216+0.6896h_{i,t-1}+0.3174[\varepsilon_{i,t-1}-0.1108]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$																	
		50	38.8	49.1	10.2	1.6	0.3	61.0	34.5	3.4	0.8	0.3	52.7	41.2	5.0	0.9	0.3
		100	12.1	72.2	13.6	1.8	0.3	28.0	66.5	4.7	0.6	0.1	17.0	71.7	9.9	1.0	0.3
		200	0.2	81.9	15.2	2.4	0.3	3.3	90.8	4.9	0.7	0.3	0.4	84.5	12.5	1.9	0.7
Model E: $\varepsilon_{i,t}=h_{i,t}^{1/2}v_{i,t}$, $h_{i,t}=0.005+0.7h_{i,t-1}+0.28[\lvert\varepsilon_{i,t-1}\rvert-0.23\varepsilon_{i,t-1}]^2$, $v_{i,t}\sim$ i.i.d. $N(0,1)$, $i=1,...,p$																	
		50	39.9	49.1	9.4	1.2	0.4	62.7	33.2	3.4	0.5	0.2	53.9	40.3	4.8	0.7	0.3
		100	11.4	75.1	11.7	1.6	0.3	26.3	68.2	4.7	0.6	0.2	16.2	74.3	8.3	0.9	0.3
		200	0.2	85.0	12.7	1.8	0.3	2.2	92.2	4.6	0.7	0.2	0.3	87.5	10.4	1.4	0.5
Model F: $\varepsilon_{i,t}=v_{i,t}\exp(h_{i,t})$, $h_{i,t}=\lambda h_{i,t-1}+0.5\xi_{i,t}$, $(\xi_{i,t},v_{i,t})\sim$ i.i.d. $N(0,\mathrm{diag}(\sigma_\xi^2,1))$, $i=1,...,p$																	
λ	σ_ξ	T															
0.936	0.424	50	28.8	49.4	16.9	3.9	1.0	55.7	37.2	5.6	1.2	0.3	41.0	45.8	10.3	2.2	0.7
		100	11.8	60.2	22.7	4.7	0.6	34.3	57.1	7.2	1.2	0.2	16.3	62.9	17.0	2.9	0.8
		200	0.5	69.2	25.2	4.5	0.7	8.6	83.5	6.7	0.9	0.3	0.8	73.8	20.8	3.7	0.9
0.951	0.314	50	32.8	49.3	14.3	2.9	0.6	57.6	36.4	4.6	1.1	0.4	45.4	43.9	8.3	1.8	0.5
		100	12.3	65.0	18.9	3.2	0.6	32.6	60.1	6.2	0.9	0.2	17.2	66.4	13.4	2.5	0.4
		200	0.3	74.4	21.7	2.8	0.7	6.1	86.9	5.8	0.9	0.4	0.8	78.5	17.5	2.5	0.7

TABLE 9

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
VAR (2) case, p = 2, True Rank is 0.

			Q-based			Q^b-based			Q^s-based		
		$r =$	0	1	2	0	1	2	0	1	2
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$											
d_0	d_1	T									
0.0	0.0	50	87.8	11.0	1.2	92.7	5.9	1.4	93.2	5.6	1.2
		100	91.1	7.9	0.9	94.0	4.8	1.2	93.8	5.1	1.0
		200	93.0	6.3	0.8	95.1	3.9	1.0	94.7	4.4	0.9
0.3	0.65	50	83.7	14.8	1.5	92.0	6.4	1.6	89.9	8.4	1.7
		100	87.1	11.5	1.4	93.3	5.6	1.1	90.6	7.8	1.6
		200	89.5	9.5	1.1	94.1	5.0	0.9	91.2	7.5	1.3
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$											
d_0	d_1	T									
0.0	0.0	50	87.7	10.9	1.4	93.8	4.8	1.4	93.3	5.5	1.2
		100	91.6	7.5	0.9	95.0	4.0	1.0	94.3	4.7	1.0
		200	92.7	6.7	0.6	94.6	4.5	0.9	94.4	4.7	0.9
0.3	0.65	50	85.7	12.6	1.6	92.8	5.5	1.7	92.0	6.7	1.3
		100	89.3	9.5	1.2	94.4	4.6	1.0	92.4	6.4	1.2
		200	91.0	8.1	0.9	94.1	5.0	0.9	92.6	6.3	1.1
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}]$ $v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$											
		50	83.6	14.5	1.9	91.7	6.7	1.6	89.7	8.5	1.8
		100	86.7	11.7	1.6	93.3	5.5	1.2	90.9	7.4	1.7
		200	89.2	9.9	0.9	94.1	5.1	0.8	91.4	7.5	1.0
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2$ $v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$											
		50	82.1	15.7	2.2	91.1	6.9	2.0	88.2	9.6	2.1
		100	84.0	14.2	1.8	92.7	6.1	1.2	87.6	10.3	2.1
		200	85.0	13.4	1.6	93.8	5.5	0.7	87.5	10.7	1.8
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2$ $v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$											
		50	83.0	15.1	2.0	91.7	6.7	1.6	89.2	9.0	1.8
		100	85.6	12.8	1.6	93.0	5.8	1.2	89.2	9.0	1.8
		200	87.1	11.6	1.3	93.6	5.5	0.8	88.9	9.7	1.4
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t}$ $(\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$											
λ	σ_ξ	T									
0.951	0.314	50	78.3	19.1	2.6	91.2	7.5	1.4	85.9	11.7	2.4
		100	80.1	17.7	2.2	91.9	7.1	1.0	84.2	13.6	2.2
		200	83.1	15.3	1.6	94.1	5.1	0.9	86.2	12.0	1.8

TABLE 10
Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
VAR (2) case, p = 3, True Rank is 0.

			Q-based				Q^b-based				Q^s-based			
		$r =$	0	1	2	3	0	1	2	3	0	1	2	3
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$														
d_0	d_1	T												
0.0	0.0	50	78.6	18.2	2.7	0.5	93.5	5.2	0.8	0.5	93.1	5.6	0.8	0.5
		100	87.5	10.8	1.4	0.3	94.4	4.5	0.8	0.3	94.1	4.9	0.8	0.2
		200	91.5	7.5	0.8	0.2	94.8	4.3	0.7	0.3	94.4	4.8	0.5	0.3
0.3	0.65	50	73.0	22.5	3.8	0.7	91.5	6.9	0.9	0.8	89.0	8.9	1.4	0.7
		100	82.7	15.0	2.0	0.3	92.9	6.1	0.8	0.2	89.9	8.7	1.2	0.2
		200	86.6	12.2	1.0	0.2	94.2	5.2	0.5	0.1	90.6	8.3	0.8	0.2
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$														
d_0	d_1	T												
0.0	0.0	50	78.0	18.6	2.8	0.6	93.1	5.6	0.8	0.5	92.4	6.4	0.8	0.3
		100	87.8	10.8	1.2	0.3	94.5	4.7	0.6	0.3	93.9	5.3	0.4	0.4
		200	91.3	7.8	0.7	0.1	94.9	4.5	0.4	0.2	94.5	4.9	0.5	0.1
0.3	0.65	50	75.4	20.8	3.1	0.7	92.0	6.5	1.0	0.5	90.4	8.0	1.1	0.5
		100	85.8	12.5	1.6	0.1	93.9	5.2	0.6	0.3	92.2	6.7	0.7	0.4
		200	89.0	9.9	0.9	0.2	94.1	5.1	0.5	0.3	92.5	6.7	0.5	0.3
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}], v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$														
		50	72.3	22.8	4.2	0.7	91.1	7.0	1.2	0.7	88.6	9.3	1.4	0.6
		100	82.0	15.8	1.9	0.3	93.0	6.1	0.7	0.3	89.4	9.2	1.2	0.2
		200	86.8	11.8	1.3	0.1	94.0	5.2	0.5	0.2	90.8	8.1	0.8	0.3
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$														
		50	70.2	24.9	4.0	0.9	90.2	8.1	1.1	0.7	87.7	10.1	1.4	0.9
		100	79.0	18.0	2.5	0.4	92.2	6.6	0.9	0.3	87.1	10.8	1.7	0.5
		200	81.2	16.7	1.8	0.3	93.3	5.9	0.6	0.2	85.6	12.3	1.7	0.5
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$														
		50	71.5	23.7	4.0	0.8	90.9	7.6	1.0	0.6	88.5	9.4	1.5	0.6
		100	80.1	17.3	2.3	0.3	92.6	6.3	0.9	0.2	87.8	10.1	1.5	0.5
		200	83.4	15.0	1.4	0.2	93.5	5.8	0.5	0.2	87.4	11.1	1.1	0.3
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t},\ (\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$														
λ	σ_ξ	T												
0.951	0.365	50	67.0	27.0	5.1	0.8	89.4	8.6	1.7	0.2	83.9	13.1	2.3	0.7
		100	73.1	22.2	4.0	0.6	91.3	7.1	1.3	0.2	82.0	14.9	2.4	0.7
		200	77.4	20.0	2.3	0.2	93.3	6.1	0.4	0.2	83.1	14.9	1.6	0.4

TABLE 11

Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank. VAR (2) case, p = 4, True Rank is 0.

			Q-based					Q^b-based					Q^s-based				
		$r=$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	58.5	31.2	8.1	1.6	0.5	91.5	7.0	1.0	0.3	0.2	91.1	7.8	0.8	0.2	0.1
		100	81.1	16.2	2.2	0.3	0.1	94.8	4.4	0.6	0.1	0.1	94.1	5.1	0.6	0.1	0.1
		200	89.1	9.7	1.0	0.2	0.0	95.2	4.0	0.6	0.1	0.1	94.9	4.4	0.5	0.1	0.1
0.3	0.65	50	53.8	35.3	8.6	1.9	0.5	90.6	7.5	1.3	0.4	0.2	88.8	9.2	1.3	0.5	0.2
		100	74.6	21.3	3.3	0.7	0.2	93.0	6.0	0.7	0.2	0.1	89.4	9.2	1.0	0.2	0.1
		200	83.9	14.1	1.7	0.3	0.0	94.3	5.0	0.5	0.1	0.1	90.6	8.2	0.8	0.2	0.2
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	58.7	31.7	7.6	1.8	0.3	92.3	6.4	1.1	0.3	0.0	91.3	7.4	1.0	0.3	0.0
		100	81.3	15.8	2.6	0.2	0.1	94.2	5.0	0.7	0.1	0.0	93.2	6.1	0.6	0.1	0.0
		200	89.1	9.7	1.1	0.1	0.0	94.8	4.6	0.5	0.1	0.0	94.2	5.2	0.4	0.1	0.1
0.3	0.365	50	55.3	33.6	8.8	1.9	0.4	91.4	7.2	1.2	0.2	0.0	89.5	8.8	1.3	0.3	0.1
		100	77.8	18.4	3.1	0.5	0.1	93.7	5.4	0.6	0.2	0.1	91.6	7.4	0.7	0.3	0.1
		200	86.5	11.8	1.3	0.3	0.0	94.8	4.5	0.6	0.1	0.0	92.7	6.4	0.7	0.1	0.0
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}], v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
		50	52.6	35.3	9.4	2.1	0.6	89.6	8.7	1.2	0.4	0.2	87.5	10.4	1.3	0.5	0.2
		100	74.5	20.9	3.7	0.8	0.1	92.4	6.0	1.2	0.2	0.1	88.6	9.7	1.3	0.3	0.1
		200	83.3	14.7	1.6	0.3	0.0	94.0	5.2	0.6	0.1	0.1	90.6	8.3	0.8	0.1	0.1
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2, v_{i,t} \sim$ i.i.d. $N(0,1), i = 1, ..., p$																	
		50	52.4	35.0	9.7	2.3	0.6	89.4	8.4	1.6	0.4	0.2	86.5	11.0	1.8	0.5	0.2
		100	70.3	24.4	4.3	0.9	0.1	92.0	6.3	1.2	0.3	0.1	86.2	11.6	1.7	0.5	0.2
		200	77.1	19.4	2.9	0.4	0.1	93.4	5.7	0.7	0.2	0.0	85.4	12.3	1.7	0.4	0.2
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2, v_{i,t} \sim$ i.i.d. $N(0,1), i = 1, ..., p$																	
		50	54.1	33.8	9.6	2.0	0.5	89.7	8.3	1.5	0.3	0.2	86.9	10.9	1.5	0.5	0.2
		100	72.6	22.5	4.1	0.6	0.2	92.6	6.1	0.9	0.2	0.1	87.6	10.3	1.5	0.4	0.2
		200	80.0	17.3	2.2	0.4	0.1	93.8	5.3	0.8	0.1	0.1	87.5	10.5	1.5	0.2	0.2
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t},\ (\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$																	
λ	σ_ξ	T															
0.951	0.365	50	47.8	36.7	12.2	2.7	0.6	86.6	10.9	1.8	0.5	0.2	80.2	15.6	3.3	0.7	0.3
		100	63.4	29.6	5.7	1.1	0.2	90.5	8.2	0.9	0.2	0.1	80.8	16.3	2.3	0.5	0.2
		200	69.8	24.6	4.7	0.7	0.1	92.7	6.1	1.0	0.2	0.1	79.3	17.2	2.9	0.5	0.2

TABLE 12
Standard and Bootstrap Sequential Procedures for Selecting the Co-Integration Rank.
VAR (2) case, p = 5, True Rank is 0.

			Q-based					Q^b-based					Q^s-based				
		$r =$	0	1	2	3	4,5	0	1	2	3	4,5	0	1	2	3	4,5
Model A: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	29.7	40.4	20.6	6.4	2.8	90.3	7.8	1.4	0.3	0.2	88.5	9.7	1.5	0.2	0.1
		100	68.0	25.1	5.4	1.2	0.3	93.5	5.6	0.8	0.1	0.1	92.6	6.3	0.8	0.1	0.1
		200	84.2	13.9	1.6	0.3	0.1	94.8	4.7	0.3	0.1	0.0	94.7	4.7	0.5	0.1	0.1
0.3	0.65	50	27.8	41.4	21.2	6.7	2.9	88.0	9.6	1.7	0.4	0.2	85.4	11.7	2.1	0.5	0.2
		100	61.9	28.8	7.4	1.6	0.3	91.4	7.2	1.0	0.3	0.1	87.5	10.4	1.7	0.3	0.1
		200	77.1	20.0	2.5	0.3	0.1	93.7	5.7	0.5	0.0	0.0	89.2	9.7	0.8	0.2	0.1
Model B: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, h_{i,t} = \omega + d_0 \varepsilon_{i,t-1}^2 + d_1 h_{i,t-1}, v_{i,t} \sim i.i.d.\ t_5, i = 1, ..., p$																	
d_0	d_1	T															
0.0	0.0	50	29.8	41.7	20.3	5.8	2.4	89.8	8.7	1.1	0.3	0.1	88.9	9.4	1.5	0.2	0.1
		100	67.7	25.9	5.1	1.0	0.2	94.3	4.9	0.5	0.2	0.0	93.5	5.7	0.6	0.1	0.0
		200	83.2	14.3	2.1	0.3	0.1	94.1	5.1	0.5	0.1	0.1	93.6	5.6	0.6	0.1	0.1
0.3	0.65	50	27.6	41.3	21.5	7.1	2.4	89.0	9.1	1.4	0.4	0.1	87.4	10.6	1.7	0.2	0.1
		100	64.6	27.9	5.8	1.4	0.3	93.4	5.6	0.9	0.2	0.0	90.7	8.0	0.9	0.3	0.1
		200	80.2	17.0	2.5	0.3	0.1	93.9	5.1	0.7	0.2	0.0	91.9	7.1	0.9	0.1	0.1
Model C: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t}, \ln(h_{i,t}) = -0.23 + 0.9 \ln(h_{i,t-1}) + 0.25[\lvert v_{i,t-1}^2 \rvert - 0.3 v_{i,t-1}], v_{i,t} \sim i.i.d.\ N(0,1), i = 1, ..., p$																	
		50	27.0	40.8	22.0	7.4	2.7	87.5	9.7	2.1	0.6	0.1	84.4	12.6	2.2	0.6	0.2
		100	60.1	30.5	7.7	1.4	0.3	91.5	6.9	1.3	0.3	0.0	87.4	10.5	1.5	0.4	0.1
		200	77.5	19.7	2.4	0.3	0.1	93.5	5.8	0.5	0.1	0.0	89.8	9.3	0.7	0.2	0.1
Model D: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.0216 + 0.6896 h_{i,t-1} + 0.3174[\varepsilon_{i,t-1} - 0.1108]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$																	
		50	26.9	40.9	22.2	7.1	2.9	87.3	9.9	2.2	0.5	0.2	84.1	12.5	2.5	0.7	0.2
		100	57.2	32.1	8.4	1.9	0.4	90.7	7.5	1.3	0.4	0.2	85.2	12.0	2.1	0.5	0.2
		200	70.3	24.8	4.3	0.5	0.2	92.6	6.7	0.6	0.1	0.0	83.8	14.2	1.7	0.2	0.1
Model E: $\varepsilon_{i,t} = h_{i,t}^{1/2} v_{i,t},\ h_{i,t} = 0.005 + 0.7 h_{i,t-1} + 0.28[\lvert \varepsilon_{i,t-1} \rvert - 0.23 \varepsilon_{i,t-1}]^2,\ v_{i,t} \sim$ i.i.d. $N(0,1),\ i = 1, ..., p$																	
		50	27.5	40.7	21.8	7.3	2.7	87.8	9.6	2.1	0.4	0.2	84.3	12.7	2.2	0.6	0.2
		100	59.1	30.9	7.8	1.8	0.4	91.8	6.8	0.9	0.5	0.1	86.7	10.9	1.7	0.4	0.2
		200	73.5	22.8	3.3	0.4	0.1	93.8	5.5	0.5	0.1	0.0	87.0	11.5	1.3	0.2	0.1
Model F: $\varepsilon_{i,t} = v_{i,t} \exp(h_{i,t}),\ h_{i,t} = \lambda h_{i,t-1} + 0.5 \xi_{i,t},\ (\xi_{i,t}, v_{i,t}) \sim$ i.i.d. $N(0, \mathrm{diag}(\sigma_\xi^2, 1)),\ i = 1, ..., p$																	
λ	σ_ξ	T															
0.951	0.365	50	24.0	40.3	23.6	9.2	3.0	83.5	13.0	2.6	0.5	0.3	77.3	17.5	4.0	0.8	0.4
		100	50.4	35.4	10.9	2.7	0.6	88.3	9.5	1.5	0.5	0.1	78.7	17.1	3.1	0.8	0.2
		200	62.8	28.3	7.1	1.5	0.3	91.9	6.6	1.2	0.3	0.1	78.6	16.6	3.6	1.0	0.2

TABLE 13
Standard and Bootstrap Co-Integration Tests.
Monthly U.S. Interest Rate Data, 1970: 1-2000: 12.

r	Eigenvalue	Q_r	Asymptotic 5% Critical Value	Asymptotic p-value	i.i.d. Bootstrap p-value	Wild Bootstrap p-value
0	0.202	193.66	75.74	0.000	0.000	0.000
1	0.152	110.42	53.42	0.000	0.000	0.000
2	0.074	49.66	34.80	0.008	0.001	0.011
3	0.048	21.24	19.99	0.037	0.034	0.103
4	0.009	3.25	9.13	0.544	0.522	0.630

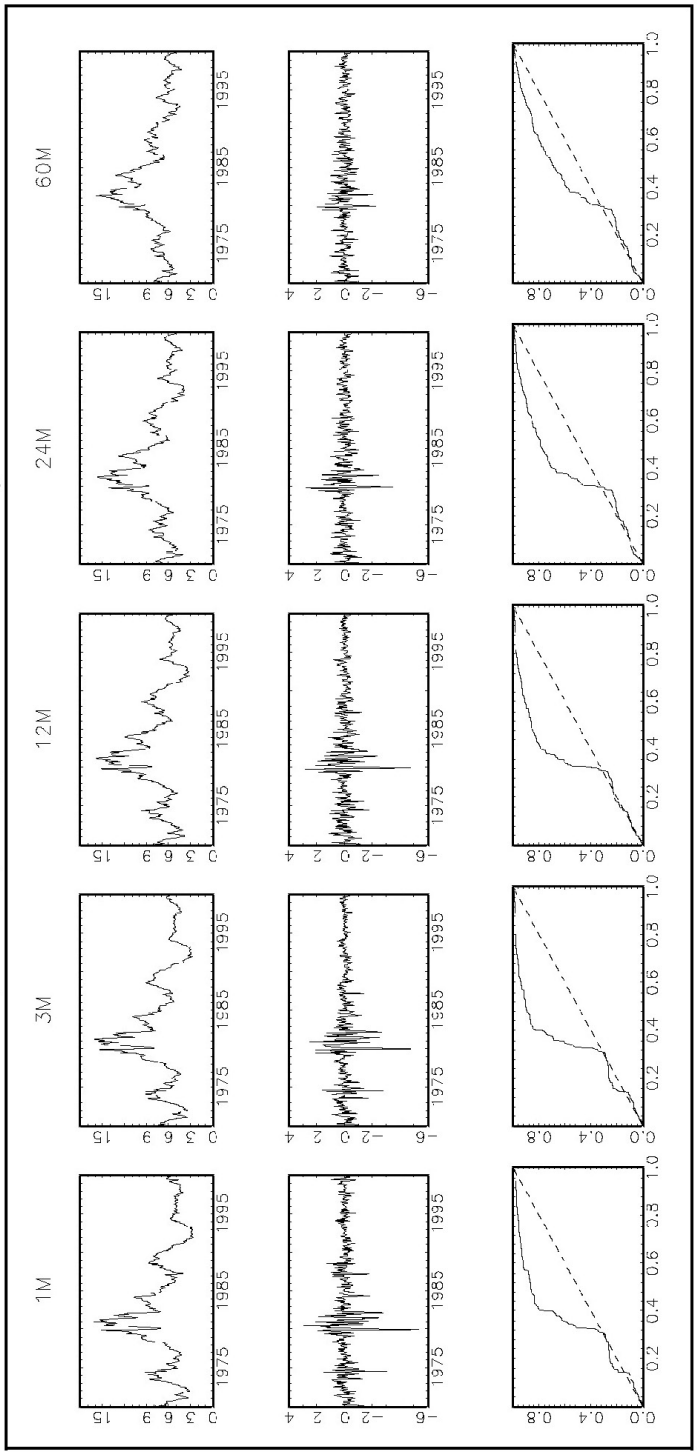

FIGURE 1
Levels, First Differences and Estimated Variance Profiles for Monthly data of U.S. Treasury Zero-Coupon Yields with One Month, Three Months, One Year, Two Years and Five Years Maturity, 1970-2000.