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A note on mixture prior distributions with applications in actuarial statistic

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ABSTRACT

The paper presents a Bayesian sensitivity analysis for the credibility theory related to the net premium principle. Thus, the mixture model in prior distribution is used for the separation of subpopulations. This construction is adapted to the usual robust Bayesian results and these are exploited to obtain lower and upper bounds for the premium. Two realistic examples illustrate the application of this method.

Keywords: Bayesian Robustness, Bimodal Distribution, Credibility Theory, Net Premium Principle, Good-risk/Bad-risk, ε-Contaminated Classes of Priors.

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RESUMEN

El artículo presenta un análisis de sensibilidad Bayesiano aplicando el principio de prima neta para el cálculo de la prima. Para separar a la población en dos colectivos, se utiliza un modelo de mixturas de distribuciones *a priori*. Con este planteamiento, se realiza un análisis Bayesiano robusto obteniendo las cotas inferiores y superiores de la prima. Por último, se ilustran los resultados obtenidos con dos ejemplos numéricos.

Palabras Clave: Análisis de sensibilidad Bayesiano, Distribución Bimodal, Teoría de la Credibilidad, Principio de Prima Neta, Buenos-Malos riesgos, Clase ε-contaminación de distribuciones *a priori.*

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1. INTRODUCTION AND MOTIVATION

The attractive features of mixed distributions are widely used for modelling heterogeneous portfolio claim counts in an actuarial application: this fact led us to develop robust Bayesian inferences for the location parameter (the premium to be charged, in credibility theory) of the sample distribution.

Credibility theory is a set of ideas concerning the systematic adjustment of insurance premiums as claims experience is obtained. One goal of credibility theory is to estimate the conditional mean $E[X|\theta]$, known as the net premium principle. The loss distribution of a given risk is, therefore, characterized by its conditional mean, but that mean is generally unknown. Therefore, we assume that the value θ is fixed for a given risk, although it is generally unknown. The probability density function of Φ is given by $\pi(\theta)$; this is the prior distribution in Bayesian analysis, also called the structure function, the distribution that represents one's uncertainty about the parameter Φ before observing claim data for a given risk. Let Φ be a random variable, and $X_i|\Theta=\theta,\ i=1,2,...,t$, the claims or loss amount in subsequent years. We assume that given θ the X_i 's are conditionally independent and identically distributed random variables.

In this paper we study the situation where the collective has two types of risks; α_1% are good risks (usually this is a high percent of the population in study) with a low claim or loss amount probability, and the other α_2% are bad risks (a low percent of the population, in practice) with a high claim or loss amount probability (see Hewitt, 1966; Hewitt and Lefkowitz, 1979 and Venter, 1991, among others), which can be modelled by two structure functions (prior distributions) $\pi_1(\theta)$ and $\pi_2(\theta)$. Therefore our prior distribution of θ is given by $\pi_0(\theta)=\sum_{i=1}^{2}\alpha_i\pi_i(\theta)$, with $\alpha_1,\alpha_2\geq 0$, $\alpha_1+\alpha_2=1$.

A premium calculation principle assigns to any risk X (with probability function $f(x\theta)$, where x takes values in the sample space X and θ is considered a realization of a parameter space Φ) a real number, which is the premium.

In the case of the net premium principle, the premium (Heilmann, 1989; Landsman and Makov, 1998; Young, 2000) is given by

$$\mathrm{P}(\theta)=E_{\mathrm{X}}[X|\theta]=\int_{\mathrm{X}} xf(x|\theta)dx,\ \ \theta\in\Theta. \qquad \textbf{(1)}$$

In ratemaking, the actuary takes a claim experience $M=m$ from the random variables $X_1,X_2,...,X_t$ and uses this information to estimate the unknown fair premium

$P(\theta)$. Now, let denote the prior density function of θ, which is given by a convex sum of two prior distributions; the good and bad risk distributions.

The posterior distribution of θ given the experience m is given by

$$\pi_0(\theta|m)=\frac{f(m|\theta)\pi_0(\theta)}{\int_\Theta f(m|\theta)\pi_0(\theta)d\theta}=\sum_{i=1}^{2}\alpha_i'\pi_i(\theta|m), \tag{2}$$

where

$$\alpha_i'=\frac{\alpha_i p(m|\pi_i)}{\sum_{i=1}^{2}\alpha_i p(m|\pi_i)}$$

and $p(m|\pi_i)=\int_\Theta f(m|\theta)\pi_i(\theta)d\theta$ is the marginal distribution of M with respect to the prior π_i.

Using the net premium principle, the Bayesian net premium (Heilmann, 1989; Eichenauer et al., 1988) is now given by

$$\mathrm{P}^*_{\pi_0}(m)=\int_\Theta \mathrm{P}(\theta)\pi_0(\theta|m)d\theta=\sum_{i=1}^{2}\alpha_i'P^*_{\pi_i}(m).$$

Our approach is based on the assumption that the practitioner is unwilling or unable to choose a functional form of the structure function, π_0 but that he may be able to restrict the possible prior to a class that is suitable for quantifying the actuary's uncertainty. Therefore it is of interest to study how the premium, $P^*_\pi(m)$, for priors in such a class behaves.

We use the classical ε-contamination class of priors (Berger, 1985,1994; Sivanganesan, 1991; Sivaganesan and Berger, 1989; Boratynska, 1996; Ying-Hsing and Ming-Chung, 1997, and Ríos and Ruggeri, 2000; among others), $\Gamma_\varepsilon=\{\pi=(1-\varepsilon)\pi_0+\varepsilon q, q\in Q\}$, where π_0 is the base elicited prior, Q is the class of allowed contaminations and $\varepsilon\in[0,1]$ measures the uncertainty of the base prior π_0. Since in our model there are two distinct claim or loss amount generating processes, where some claims or losses are regular and may be described by a p.d.f. $\pi_1(\theta)$, while others are nuisance high claims or losses which may be described by a p.d.f. $\pi_2(\theta)$, our -contamination class is given by

$$\Gamma_\varepsilon^j=\left\{\pi=(1-\varepsilon)\sum_{i=1}^{2}\alpha_i\pi_i+\varepsilon\sum_{i=1}^{2}\beta_i q_i,\ q_i\in Q^j\right\},\quad j=1,2,\quad \beta_1,\beta_2\geq 0,\ \beta_1+\beta_2=1.$$

For $Q^1 = \{\text{All probability distributions}\}$ we determine the range of Bayesian net premiums as π varies over Γ_ε^j. Now, if we want the model to include distributions with similar shapes to the prior distributions, we can consider the contamination class

$$Q^2 = \left\{ \begin{matrix} q_i(\theta) : q_i \text{ to be unimodal with the same} \\ \text{mode, } \theta_i, \text{ as that of } \pi_i \end{matrix} \right\}.$$

2. LOWER AND UPPER BOUNDS IN A ROBUST BAYESIAN ANALYSIS WITH MIXTURE PRIORS

In this paper the range of Bayesian net premiums is found over the class

$$\Gamma_\varepsilon^j = \left\{ \pi = (1-\varepsilon)\sum_{i=1}^{2} \alpha_i \pi_i + \varepsilon \sum_{i=1}^{2} \beta_i q_i,\ q_i \in Q^j \right\}, \quad (j = 1,2).$$

with

$$Q^1 = \{\text{All probability distributions}\},$$

and

$$Q^2 = \left\{ \begin{matrix} q_i(\theta) : q_i \text{ is unimodal with the same} \\ \text{mode, } \theta_i, \text{ as that of } \pi_i \end{matrix} \right\},$$

where θ_i is the modal value for the distribution of expected value of claims (or loss amount), $\pi_i(\theta)$.

Using Γ_ε^1, if similar conclusions are obtained, no additional information is required; however, if conclusions differ markedly, we must obtain more information. In this case we could acquire partial information about the prior (for example, the bimodality) and consider all prior distributions that are compatible with this information, using Γ_ε^2.

It is straightforward to rewrite the Bayesian premium under Γ_ε^1 class as

$$P_\pi^*(m) = \frac{(1-\varepsilon)\left\{\sum_{i=1}^{2} \alpha_i p(m|\pi_i)\right\} P_{\pi_0}(m) + \varepsilon \int_\Theta g(\theta) f(m|\theta) q(\theta) d\theta}{(1-\varepsilon)\left\{\sum_{i=1}^{2} \alpha_i p(m|\pi_i)\right\} + \varepsilon \int_\Theta f(m|\theta) q(\theta) d\theta}, \qquad \textbf{(3)}$$

and

$$P_{\pi}^{*}(m)=\frac{(1-\varepsilon)\left\{\sum_{i=1}^{2}\alpha_i p(m|\pi_i)\right\}P_{\pi_0}(m)+\varepsilon\sum_{i=1}^{2}\beta_i\int_0^{\infty}H^{q_i}(z_i)dF(z_i)}{(1-\varepsilon)\left\{\sum_{i=1}^{2}\alpha_i p(m|\pi_i)\right\}+\varepsilon\sum_{i=1}^{2}\beta_i\int_0^{\infty}H^{i}(z_i)dF(z_i)}, \qquad \textbf{(4)}$$

under Γ_{ε}^{2} class. Now, we can easily obtain the ranges for Bayesian premiums using the following theorem (Sivaganesan and Berger, 1989; Berger and Moreno, 1994).

Theorem 1 Suppose $B>0$ and $f_i(x_i), g_i(x_i), i=1,2$ are continuous functions with $g_i(x_i)\geq 0,$ then

$$\sup_{dF_1,dF_2}\frac{A+\sum_{i=1}^{2}\int f_i(x_i)dF_i(x_i)}{B+\sum_{i=1}^{2}\int g_i(x_i)dF_i(x_i)}=\sup_{x_1,x_2}\frac{A+\sum_{i=1}^{2}f_i(x_i)}{B+\sum_{i=1}^{2}g_i(x_i)}$$

The same result holds with *sup* replaced by *inf.*

2.1. Two standard models

The most useful probability models developed in the literature representing the distribution of the number of claims in an insurance portfolio (Lemaire, 1995) are the following. The pair likelihood and structure function chosen is termed "model". The most frequently used likelihoods are Poisson (Wilmot, 1993) and Negative binomial (Lemaire, 1995). These likelihood functions are combined with structure functions like as Gamma and Inverse Gaussian, among others (Lemaire, 1992). In this paper, we present the two most useful and standard (conjugate) parametric models.

Assume that the number of claims generated annually depends upon chance, while the amount of the individual claim is taken as fixed. Suppose that the number of claims follows a Poisson distribution with parameter $\theta>0$,

$$P\{N=n\}=\frac{\theta^n e^{-\theta}}{n!}, n=0,1,\ldots,$$

and the prior density of θ is a mixture: $\pi_0(\theta) = \alpha_1 \mathrm{G}(a_1, b_1) + \alpha_2 \mathrm{G}(a_2, b_2)$,

where a_1, a_2, b_1, b_2 are positive hyperparameters, and G represents the gamma density function, i. e.:

$$\pi(\theta) \propto \theta^{a-1} e^{-b\theta}, \quad \theta > 0.$$

We call Poisson-Gamma model to this specification. Observe that modelization assumes that the risks are independent, so we take a risk parameter θ and assume that the number of claims for each policy-holder fit a Poisson, whose parameter θ varies from one individual to another, reflecting the individual's claim propensity. Examples of papers using the simple Poisson-Gamma include Eichenauer et al., 1988; Gómez et al., 1999 and Gómez et al., 2000, among others.

Another common model in credibility literature, consists of assuming an exponential distribution for the likelihood function (Heilmann, 1989), that is $f(x) = \theta e^{-\theta x}$, $x > 0$ $(\theta > 0)$, and also of considering a mixture of two gammas in the structure function (prior density).

Following results give us the methodology to compute upper and lower bounds for the premiums. These results are very important because the computing problem becomes in terms of a variable.

Proposition 1 In the indifference setting, i.e. $\pi \in \Gamma_{\varepsilon}^{1}$, the lower (upper) bound for the Bayesian net premium is given by

$$\inf_{\theta \in \Theta}(\sup) \frac{R_1 P_{\pi_0}^{*}(m) + R_2(\theta)}{R_1 + R_3(\theta)},$$

where:

(i) in the Poisson-Gamma case,

$$\mathrm{R}_1 = (1-\varepsilon) \sum_{i=1}^{2} \alpha_i \frac{a_i^{b_i}}{(b_i - 1)!} \frac{(b_i + tm - 1)!}{(a_i + t)^{b_i + tm}}, \qquad \mathrm{R}_2(\theta) = \varepsilon \theta^{tm+1} e^{-tm},$$

$$\mathrm{R}_3(\theta) = \mathrm{R}_2(\theta)/\theta \text{ and } \mathrm{P}_{\pi_0}^{*}(m) = \sum_{i=1}^{2} \alpha_i' \frac{b_i + tm}{a_i + t}.$$

(ii) in the Exponential-Gamma case,

$$\mathrm{R}_1 = (1-\varepsilon)\sum_{i=1}^{2}\alpha_i \frac{a_i^{\,b_i}}{(b-1)!}\frac{(b_i+t-1)!}{(a_i+tm)^{b_i+t}}, \quad \mathrm{R}_2(\theta) = \varepsilon\theta^{n-1}e^{-tm\theta},,$$

$$\mathrm{R}_3(\theta) = \theta \mathrm{R}_2(\theta) \text{ and } \mathrm{P}^*_{\pi_0}(m) = \sum_{i=1}^{2}\alpha_i' \frac{a_i+tm}{b_i+t-1}.$$

Proof. The proof follows from (3) and Theorem 1 with $f_2(x_2) = g_2(x_2) = 0.$

***Proposition* 2** In the bimodality setting, i.e. $\pi \in \Gamma^2_\varepsilon,$ the lower (upper) bound for the Bayesian net premium is given by $\inf_{z_1,z_2\geq 0}(\sup)\mathrm{R}(z_1,z_2),$ being

$$\mathrm{R}(z_1,z_2) = \frac{\mathrm{R}_1\mathrm{P}^*_{\pi_0}(m) + \sum_{i=1}^{2}(1/z_i)\beta_i\int_{\theta_i}^{\theta_i+z_i}\mathrm{R}_2(\theta)d\theta}{\mathrm{R}_1 + \sum_{i=1}^{2}(1/z_i)\beta_i\int_{\theta_i}^{\theta_i+z_i}\mathrm{R}_3(\theta)d\theta}, \text{ if } z_1,z_2>0,$$

$$\mathrm{R}(z_1,0) = \frac{\mathrm{R}_1\mathrm{P}^*_{\pi_0}(m) + (1/z_1)\beta_1\int_{\theta_1}^{\theta_1+z_1}\mathrm{R}_2(\theta)d\theta + \beta_2\mathrm{R}_2(\theta_2)}{\mathrm{R}_1 + (1/z_1)\beta_1\int_{\theta_1}^{\theta_1+z_1}\mathrm{R}_3(\theta)d\theta + \beta_2\mathrm{R}_3(\theta_2)}, \text{ if } z_1>0,$$

$$\mathrm{R}(0,z_2) = \frac{\mathrm{R}_1\mathrm{P}^*_{\pi_0}(m) + \beta_1\mathrm{R}_2(\theta_2) + (1/z_2)\beta_2\int_{\theta_2}^{\theta_2+z_2}\mathrm{R}_2(\theta)d\theta}{\mathrm{R}_1 + \beta_1\mathrm{R}_3(\theta_1) + (1/z_2)\beta_2\int_{\theta_2}^{\theta_2+z_2}\mathrm{R}_3(\theta)d\theta}, \text{ if } z_2>0,$$

and

$$\mathrm{R}(0,0) = \frac{\mathrm{R}_1\mathrm{P}^*_{\pi_0}(m) + \sum_{i=1}^{2}\beta_i\mathrm{R}_2(\theta_i)}{\mathrm{R}_3(\theta) + \sum_{i=1}^{2}\beta_i\mathrm{R}_3(\theta_i)},$$

where $\mathrm{R}_1, \mathrm{R}_2(\theta), \mathrm{R}_3(\theta)$ and $\mathrm{P}^*_{\pi_0}(m)$ are as in Proposition 1.

Proof. The proof follows from (4) and Theorem 1.

2.2. Other models

Previous results are only for those models and for the net premium principle. Obviously, there are other models and principles in the actuarial literature. For example, the variance principle is often used.

Using the variante Premium principle, a bonus malus system is defined by the relativities:

$$P_{BM}^{*}(m)=100\frac{E_{\pi(\theta|m)}\left[P(\theta)^2\right]}{E_{\pi(\theta|m)}\left[P(\theta)\right]}\frac{E_{\pi(\theta)}\left[P(\theta)\right]}{E_{\pi(\theta)}\left[P(\theta)^2\right]}.$$

So, robustness problem of the premiums is essential to choose the class of distributions (Eichenauer et al., 1988). In this sense, two preceeding classes are "more artifficial" and are used to solve computing problems instead of natural causes. We present now results for classes of prior moments conditions. In other contexts, this class has been studied by Eichenauer et. al (1988).

$$\mathcal{Q}_1^{*}=\left\{q:\int_{\Theta}(\theta+1)^i q(\theta)d\theta=\int_{\Theta}(\theta+1)^i\pi_0(\theta)d\theta=\alpha_i,i=1,2\right\}$$

Corollary 1: For $q\in\mathcal{Q}_1^{*}$, the upper bound for the Bayesian premium for the variance principle in the Poisson-Gamma model is given by

$$\sup_{q}P_{BM}^{*}(m)=100\frac{b(a+b)}{a(a+1)+2ab+b^2}\sup_{\theta\in\Theta}\frac{R_1(\theta)+R_2(\theta)}{R_1(\theta)+R_3(\theta)},$$

where

$$R_1(\theta)=\left(\frac{b}{b+t}\right)^2\frac{(a+tm)(a+tm+1)+(b+t)^2+2(a+tm)(b+t)}{a(a+1)+b^2+2ab}R_3(\theta),$$

$$R_2(\theta)=(\theta+1)R_4(\theta),$$

$$R_3(\theta)=(1-\varepsilon)b^a(b+t)^2\Gamma(a+tm)\left(a(a+1)+b^2+2ab\right)\left(\theta^2+3\theta+2\right)e^{t\theta}\theta^{-tm},$$

$$R_4(\theta)=\varepsilon\Gamma(a)(b+t)^{a+tm+2}\left(2(a+b)^2+a\right)(\theta+1).$$

The lower bound is obtained by replacing sup *with* inf.

Proof. The result follows from the expressions

$$p(m|\pi_0)=b^2\Gamma(a+tm)\Big/\left(\Gamma(a)(b+t)^{a+tm}\right),$$

$$\alpha_1 = E_{\pi_0}\left[\theta+1\right] = (a+b)/b,\ \alpha_2 = E_{\pi_0}\left[(\theta+1)^2\right] = \left(a(a+1)+b^2+2ab\right)/b^2$$

and considering the Lemmas 3.2.1. and A.1 in Sivaganesan and Berger (1989).

Finally, the result below gives us the relativity range when we use the class

$$\mathsf{Q}_2^* = \left\{ \begin{array}{l} \mathrm{q}:\int_\Theta \mathrm{H}_\mathrm{i}(\theta)\, q(\theta)\, d\theta = \int_\Theta \mathrm{H}_\mathrm{i}(\theta)\, \pi_0(\theta)\, d\theta = \alpha_i,\ \ i=1,\ldots,n, \\ \text{and mode of } q(\theta|m) = \text{mode of } \pi_0(\theta|m) = \lambda_0 \end{array} \right\}$$

Corollary 2: For $\mathsf{q} \in \mathsf{Q}_2^*$ the upper bound for the Bayesian premium for the variance principle in the Poisson—Gamma model is given by

$$\sup_{\mathsf{q}} P_{BM}^*(m) = \begin{cases} 100\dfrac{b(a+b)}{a(a+1)+2ab+b^2} \sup\limits_{z>0} \dfrac{\int_{\theta_0}^{\theta_0+z}\left(R_1(\theta)+R_2(\theta)\right)d\theta}{\int_{\theta_0}^{\theta_0+z}\left(R_3(\theta)+R_4(\theta)\right)d\theta} \\ 100\dfrac{b(a+b)}{a(a+1)+2ab+b^2}\dfrac{R_1(\theta_0)+R_2(\theta_0)}{R_3(\theta_0)+R_4(\theta_0)}, \qquad z=0. \end{cases}$$

with $R_1(\theta), R_2(\theta), R_3(\theta)$ and $R_4(\theta)$ as in Corollary 1.

The lower bound is obtained by replacing sup with inf.

Proof. *The proof is a consequence of applying Lemmas 3.2.1. and A.1 in Sivaganesan and Berger (1989).*

3. NUMERICAL ILLUSTRATIONS

In order to illustrate the above ideas, two numerical illustrations are given. We shall use $\beta_i = \alpha_i,\ i = 1,2.$ We have also included a measure the magnitudes of which do not depend on the premium measurement units, namely the relative sensitivity RS (see Sivaganesan, 1991; Gómez et al., 1999 and Gómez et al., 2000) which is given by

$$\mathrm{RS}^j = \frac{1}{2\mathrm{P}_{\pi_0}^*(m)}\left[\sup_{\pi\in\Gamma_\varepsilon^j} \mathrm{P}_{\pi_0}^*(m) - \inf_{\pi\in\Gamma_\varepsilon^j} \mathrm{P}_{\pi_0}^*(m)\right]\times 100\%, \quad (j=1,2).$$

Results for the variance principle analyzed in previous section are initial and we have studied the case for the net premium principle.

Example 1. Let $X|\theta$ have a Poisson distribution with parameter θ. Assume that the actuary knows that two modal values are possible and they are around 1.5 and 10 (i.e. $\theta_1 = 1.5$ and $\theta_2 = 10$), and that claims larger than 5 are less frequent than smaller claims. Thus, a plausible mixture prior density is,

$$\pi_0(\theta) = 0.8 \cdot \mathrm{G}(2,4) + 0.2 \cdot \mathrm{G}(3,30).$$

Table 1. Contains the standard Bayesian premium for three observed sample realizations. This particular situation corresponds to , i.e. no errors in the elicitation process

Table 1. Poisson-Gamma model

m	$\mathrm{P}^*_{\pi_1}(m)$	$\mathrm{P}^*_{\pi_2}(m)$	α_1'	α_2'	$\mathrm{P}^*_{\pi}(m) = \sum_{i=1}^{2} \alpha_i' \mathrm{P}^*_{\pi_i}(m)$
4	3.666	5.384	0.998	0.002	3.670
7	6.166	7.692	0.07	0.93	7.585
12	10.333	11.538	0.001	0.999	11.538

The infima and the suprema of the Bayesian net premium can be calculated as described in the previous section. The bounds of the Bayesian premium are given in Figure 1 for the classes Γ_ε^1 and Γ_ε^2. Figure 2 shows the RS factor. Furthermore, if $\alpha_2 = \beta_2 = 0$ we are in the simple unimodality setting, as in Gómez et al. (1999), Gómez et al. (2000) and Sivaganesan and Berger (1989).

Example 2. Let $X|\theta$ have an exponential distribution with parameter θ and $\pi_0(\theta) = 0.6 \cdot \mathrm{G}(60,11) + 0.4 \cdot \mathrm{G}(10,6)$. Table 2 presents the standard Bayesian premium in three sample situations.

Table 2. Exponential-Gamma model

m	$\mathrm{P}^*_{\pi_1}(m)$	$\mathrm{P}^*_{\pi_2}(m)$	α_1'	α_2'	$\mathrm{P}^*_{\pi}(m) = \sum_{i=1}^{2} \alpha_i' \mathrm{P}^*_{\pi_i}(m)$
2	2	4	0.864	0.136	2.270
4	3.333	5	0.164	0.836	4.727
5	4	5.5	0.073	0.927	5.391

Figure 1: Poisson-Gamma model. Ranges of Bayesian premiums (m=4 and m=7, above, left and right, respectively; below, left, m=12)

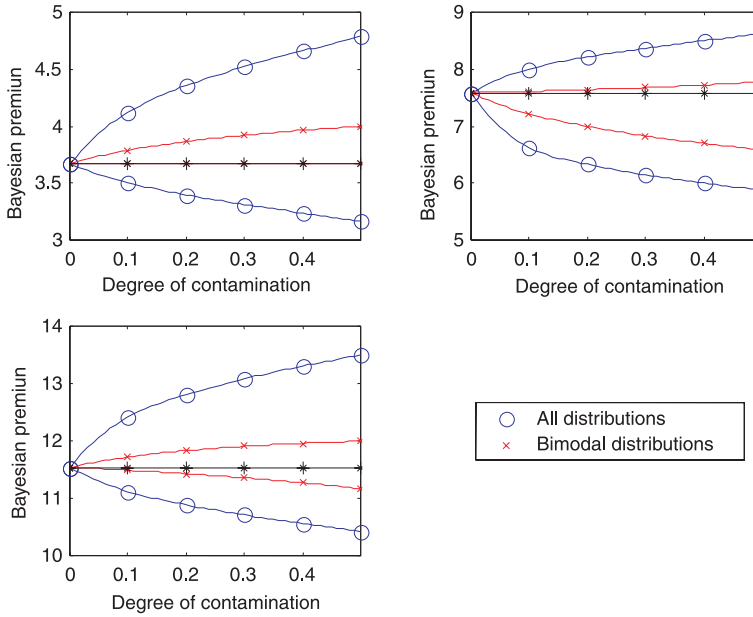
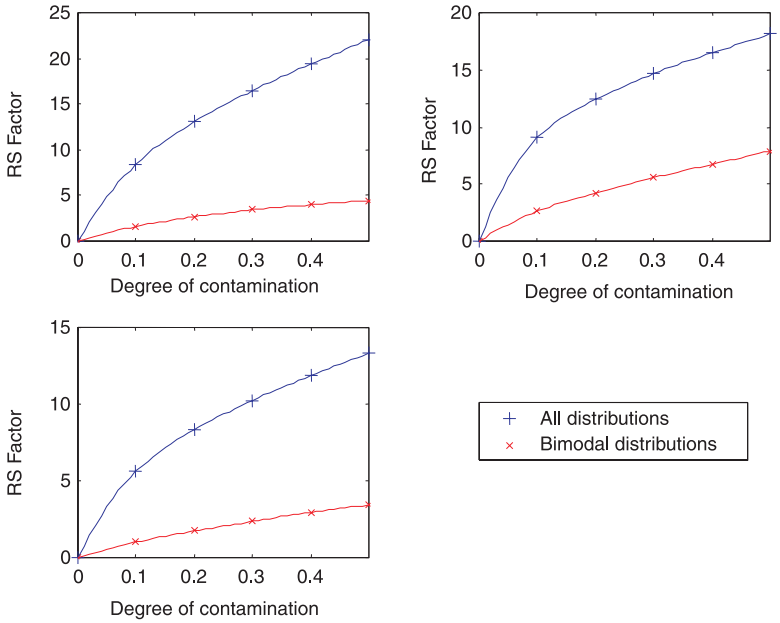
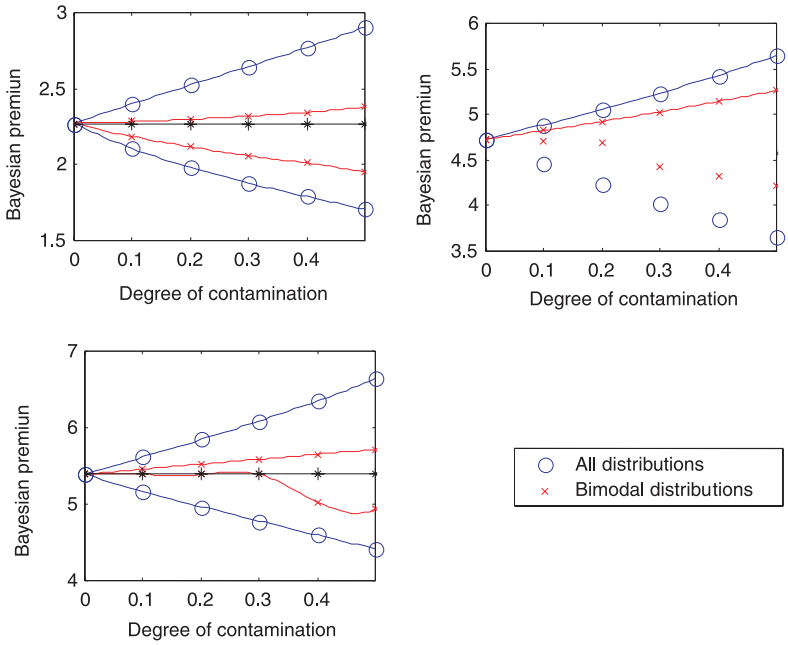


Figure 2: Poisson-Gamma model. RS factor (m=4 and m=7, above, left and right, respectively; below, left, m=12)



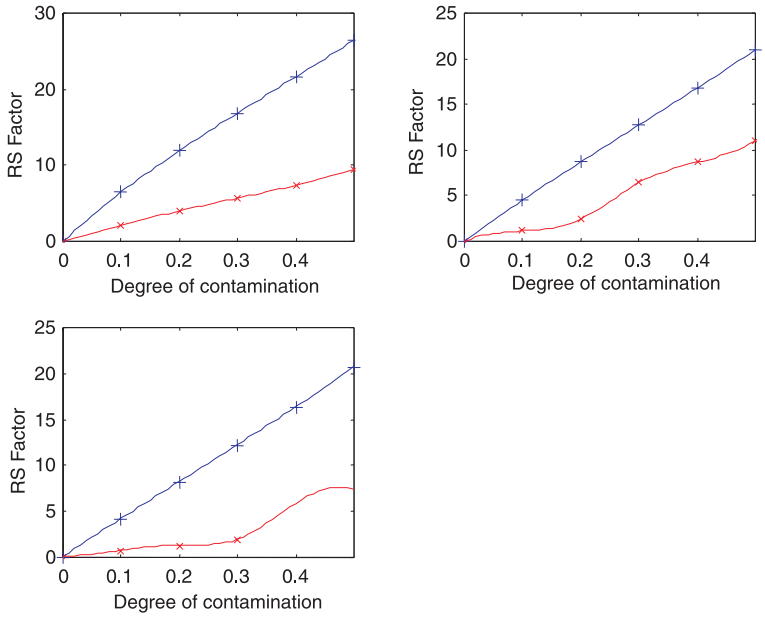
Figures 3 and 4 show the bounds of the Bayesian premium and the RS factor, respectively.

Figure 3: Exponential-Gamma model. Ranges of Bayesian premium (m=2 and m=4, above, left and right, respectively; below, left, m=5)



In both examples, the most robust situation occurs when the difference between the Bayesian net premiums for both good and bad risks are higher (m=12 in the Poisson case and m=5 in the exponential case). On the other hand, the less robust situation is in the case m=4 and m=2 for the two examples considered, when there is a higher difference between the Bayesian net premiums for both good and bad risks. Reading the figures in cases of Bayesian robustness is similar to reading a standard bonus-malus table, but taking into account that instead of a single premium, we obtain a range of premiums over the different classes considered. For instance, when uncertainty is low (of the order of 10%, epsilon=0.1) we find that in the case of Poisson-Gamma model we have a variation range of (3, 4.5). Obviously, this interval contains the value …., in table 1, which was obtained under standard Bayesian analysis.

Figure 4: Exponential-Gamma model. RS factor (m=2 and m=4, above, left and right, respectively; below, left, m=5)



4. DISCUSSION AND CONCLUSIONS

A basic assumption of credibility theory is that the values of the parameters of the probability distribution of loss are unknown. In this case, the company charges the Bayesian premium, which requires the decision maker, the actuary, to define a probability distribution for the values of the unknown parameters of this loss process, the prior distribution. Nevertheless, there are clearly many prior distributions other than which are equally compatible, and hence which could be used instead of π_0. This is justified in our model, where the prior π_0 is given by the convex sum of two prior distributions π_1 and π_2. This leads to the question of Bayesian robustness, which is treated in this paper using the ε-contamination class. Bimodality effects are very important in modelling subjective beliefs about risk parameters when this is necessary.

Standard Bayesian models in actuarial science have used conjugate models (Poisson-Gamma, Exponential-Gamma). In this paper, we present recent techniques to analyze the bimodal form of the premiums. If results are robust, the structure function is accepted. These premiums will be relatively equal if they represent the actuary´s system beliefs. However, when the model present a lack of robustness or is very sensible to the structure function, the actuary must be very carefull. Maybe, the actuary must assume another probabilistic model more flexible.

In this context, conjugate modelling can be rejected. Other non-conjugate modelling methods are possible, using recent developments in Markov chain Monte Carlo (MCMC) methods to facilitate the exploration of a posteriori actuarial magnitudes (see Makov et al., 1996 and Scollnik, 1995 and 2001, Ntzoufras and Dellaportas, 2002). Thus, log-normal model presented in Nzoufras and Dellaportas (2002) developed for the log-adjusted claim amounts can be easily implemented using Gibbs sampling.

Finally, all the theorems and results discussed in this article can be used for other premium calculation principles (Heilmann, 1989; Gómez et al., 1999 and Gómez et al., 2000), such as exponential, Esscher and variance, among others.

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